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Higher order moments of the estimated tangency portfolio weights

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Abstract

In this paper we consider the estimated weights of tangency portfolio. The returns are assumed to be independently and multivariate normally distributed. We derive analytical expressions for the higher order non-central and central moments of these weights. Moreover, the expressions for mean, variance, skewness and kurtosis of the estimated weights are obtained in closed-forms. Finally, we complement our result with an empirical study where we analyze a portfolio with actual returns of eight financial indexes listed in NASDAQ stock exchange.

JEL Classification: C10, C44

ASM Classification: 62H99, 91G10

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1 Introduction

The fundamental goal of the portfolio theory as devised by Markowitz (1952) is to determine efficient way of portfolio allocation. The mean-variance optimization technique plays a central role in allocating the investments among different asset. According to it, the investor allocates the wealth among risky assets by maximizing the expected return based on a given level of risk or by minimizing the risk for a given level of expected returns. The trade-off between the risk and return of the portfolio is at the heart of portfolio theory, which seeks to find optimal allocations of the investor's initial wealth to the available assets. The tangency portfolio is the portfolio that consists of both risky and risk-free assets. In order to have entire understanding of the conditions and processes in a portfolio, the study on statistical properties of the tangency portfolio is crucial and unavoidable.

Statistical properties of the estimated tangency portfolio weights are intensively discussed in the literature. Britten-Jones (1999) developed an F-test for the tangency portfolio weights, while Bodnar (2009) delivered sequential monitoring procedures for the tangency portfolio weights. The univariate density of the tangency portfolio as well as its asymptotic distribution under the assumption of independently and multivariate normally distributed returns are obtained by Okhrin and Schmid (2006). Later on, Bodnar and Okhrin (2011) derived the explicit density of the linear transformation of the estimated weights and suggested several exact tests of general linear hypothesis about the elements of the portfolio weights. Kotsiuba and Mazur (2015) derived the approximate density function formula for the weights which is based on the Gaussian integral and the third order Taylor expansion. Bodnar et al. (2017) extended the results by Bodnar and Okhrin (2011) in the setting when both the population and the sample covariance matrices are singular. Moreover, they established the high-dimensional asymptotic distribution of the estimated weights of the tangency portfolio when both the portfolio dimension and the sample size increase to infinity. Finally, Bauder et al. (2017) studied the distributional properties of the weights of the tangency portfolio from the Bayesian point of view.

The aim of the present paper is to derive the higher order moments of the sample weights of the tangency portfolio in closed-forms when the returns are assumed to be independently and multivariate normally distributed. As argued by Okhrin and Schmid (2006), the knowledge of portfolio weights leads to information about the expected portfolio return and the variance of the portfolio return. Since the expected portfolio returns play a crucial role in most financial theories, the knowledge of higher moments of the estimated portfolio weights can be helpful in learning about the expected portfolio return as well as the portfolio variance. Similarly, via deriving expressions for skewness and kurtosis of estimated weights, we would be able to understand the tail and asymmetric behaviour of wealth distribution. Moreover, Okhrin and Schmid (2006) shows that the moments of the optimal portfolio weights are very sensitive to changes in the moments of stock returns. The obtained expressions for the higher moments of the estimated portfolio weights can, therefore, be very informative to account for tail risks in making portfolio strategies. We have obtained explicit expressions for partial cases such as the mean, variance, skewness, and kurtosis. In particular, we take a look at the skewness and kurtosis for measuring the deviation from the normal distribution.

This paper is organized as follows. In Section 2, we establish preliminary results which we use in proving main results in Section 3. In Section 3, we deliver explicit formulas for the higher order non-central and central moments of the estimated tangency portfolio weights. Moreover, we derive the mean, variance, skewness and kurtosis in closed-forms. The results of the empirical study are given in Section 4, while Section 5 summarizes the paper.

2 Preliminary Results

In this section, we present the preliminary results, which are used in proving our main results of Section 3. Let us note that our results are complementary to the ones obtained in Bodnar and Okhrin (2011), Kotsiuba and Mazur (2015) and Bodnar et al. (2016).

Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$, i.e., the random matrix $\mathbf{A}$ has a k-dimensional Wishart distribution with n degrees of freedom and a positive definite covariance matrix $\mathbf{\Sigma}$. We assume that $n > k$, implying that the matrix $\mathbf{A}$ is non-singular. Also, let $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$, it means that the random vector $\mathbf{z}$ has the k-dimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\lambda\mathbf{\Sigma}$, where $\lambda > 0$ is a constant.

In Lemma 1, we present a stochastic representation for $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$, where $\mathbf{l}$ is a k-dimensional vector of constants. Let us note that this result can be found in Bodnar and Okhrin (2011). Below, the symbol $\mathcal{F}(d_1, d_2, s)$ stands for the non-central $\mathcal{F}$-distribution with d_1 and d_2 degrees of freedom and the non-centrality parameter s, while the symbol $\stackrel{d}{=}$ stands for the equality in distribution

Lemma 1. *Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$, $n > k$ and $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$ with $\lambda > 0$ and positive definite $\mathbf{\Sigma}$. Furthermore, let $\mathbf{A}$ and $\mathbf{z}$ be independent and $\mathbf{l}$ be a k-dimensional vector of constants. Then the stochastic representation of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by*

$$\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} \stackrel{d}{=} \frac{1}{u_1}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \sqrt{\left(\lambda + \frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}\,u_2\right) \qquad (1)$$

where $u_1 \sim \chi^2_{n-k+1}$, $u_2 \sim \mathcal{N}(0,1)$ and $u_3 \sim \mathcal{F}(k-1, n-k+2, s)$ with $s = \boldsymbol{\mu}^T\mathbf{R}_\mathbf{l}\boldsymbol{\mu}/\lambda$ and $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$. The random variables u_1, u_2 and u_3 are mutually independently distributed.

From Lemma 1 we get that a stochastic representation of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given in terms of independently distributed χ^2, a standard normal and a non-central $\mathcal{F}$ random variables. Let us recall that a confluent hypergeometric function ${}_1F_1(a;b;x)$ is defined by

$$
{}_1F_1(a;b;x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}, \tag{2}
$$

where $(a)_k$ and $(b)_k$ are Pochhammer symbols (cf. Abramowitz and Stegun (1984, Chapter 13)).

In the next theorem, we consider the higher order moments of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$.

Theorem 1. *Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$, $n > k$ and $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$ with $\lambda > 0$ and positive definite $\mathbf{\Sigma}$. Furthermore, let $\mathbf{A}$ and $\mathbf{z}$ be independent and $\mathbf{l}$ be a k-dimensional vector of constants. Then the r-th order moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by*

$$
\begin{aligned}
\mathbb{E}\left[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z})^r\right] &= \frac{1}{(n-k-1)...(n-k-2r+1)} \\
&\times \left[(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^r + \sum_{j=1}^{\lfloor r/2 \rfloor} \binom{r}{2j} \frac{(2j)!}{2^j j!} \left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{r-2j} \left(\lambda \mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^j \left(1 + \sum_{m=1}^{j} \binom{j}{m} c_m\right)\right]
\end{aligned} \tag{3}
$$

for $n - k + 1 > 2r$ with

$$
c_m = \frac{(k-1+2(m-1))\dots(k-1)}{(n-k-2(m-1))\dots(n-k)} e^{-\frac{s}{2}}\, {}_1F_1\left(m + \frac{k-1}{2}; \frac{k-1}{2}; \frac{s}{2}\right),
$$

where $s = \boldsymbol{\mu}^T\mathbf{R}_\mathbf{l}\boldsymbol{\mu}/\lambda$ and $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$.

Proof. From Lemma 1 we obtain that

$$
\begin{aligned}
\mathbb{E}\left[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z})^r\right] &= \mathbb{E}\left[\left(\frac{1}{u_1}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \sqrt{\left(\lambda + \frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}\,u_2\right)\right)^r\right] \\
&= \mathbb{E}\left[\left(\frac{1}{u_1}\right)^r\right]\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \sqrt{\left(\lambda + \frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}\,u_2\right)^r\right],
\end{aligned}
$$

where the last equality follows from the fact that u_1 is independent of u_2 and u_3.

Since $u_1 \sim \chi^2_{n-k+1}$, we get that $1/u_1 \sim Inv - \chi^2_{n-k+1}$. From Glen (2017, p. 18) it follows that

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{u_1}\right)^r\right] &= \frac{\Gamma\left(\frac{n-k+1}{2} - r\right)}{2^r\Gamma\left(\frac{n-k+1}{2}\right)} = \frac{\Gamma\left(\frac{n-k+1}{2} - r\right)}{2^r(\frac{n-k+1}{2} - 1)...(\frac{n-k+1}{2} - r)\Gamma\left(\frac{n-k+1}{2} - r\right)} \\
&= \frac{1}{(n-k-1)...(n-k-2r+1)}, \qquad n-k+1 > 2r.
\end{aligned}
$$

This result follows from the property $\Gamma(x) = (x-1)\Gamma(x-1)$.

Using the well-known binomial formula (see Biggs (1979, p. 129)) and the fact that u_2

and u_3 are independent, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}+\sqrt{\left(\lambda+\frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}u_2\right)^r\right]\\
=\;&\mathbb{E}\left[\sum_{i=0}^{r}\binom{r}{i}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{r-i}\left(\sqrt{\left(\lambda+\frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}u_2\right)^i\right]\\
=\;&\sum_{i=0}^{r}\binom{r}{i}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{r-i}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{i/2}\mathbb{E}\left[u_2^i\right]\mathbb{E}\left[\left(\lambda+\frac{\lambda(k-1)}{n-k+2}u_3\right)^{i/2}\right].
\end{aligned}
$$

Let us note that the odd moments of the standard normal distribution are equal to zero, i.e. $\mathbb{E}\left[u_2^{2j+1}\right]=0$ for $j\in\{0,1,2,\dots\}$, while the even moments are given by

$$
\mathbb{E}[u_2^{2j}]=\frac{(2j)!}{2^j j!}\quad\text{for}\quad j\geq 1,
$$

c.f. Walck (1996, Chapter 34.2)). It leads us to

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}+\sqrt{\left(\lambda+\frac{\lambda(k-1)}{n-k+2}u_3\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}u_2\right)^r\right]\\
=\;&\sum_{i=0}^{r}\binom{r}{i}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{r-i}\left(\lambda\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{i/2}\mathbb{E}\left[u_2^i\right]\mathbb{E}\left[\left(1+\frac{k-1}{n-k+2}u_3\right)^{i/2}\right]\\
=\;&\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^r+\sum_{j=1}^{\lfloor r/2\rfloor}\binom{r}{2j}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{r-2j}\left(\lambda\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{j}\mathbb{E}\left[u_2^{2j}\right]\mathbb{E}\left[\left(1+\frac{k-1}{n-k+2}u_3\right)^{j}\right]\\
=\;&\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^r+\sum_{j=1}^{\lfloor r/2\rfloor}\binom{r}{2j}\frac{(2j)!}{2^j j!}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{r-2j}\left(\lambda\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{j}\mathbb{E}\left[\left(1+\frac{k-1}{n-k+2}u_3\right)^{j}\right].
\end{aligned}
$$

Applying the binomial formula again we get

$$
\begin{aligned}
\mathbb{E}\left[\left(1+\frac{k-1}{n-k+2}u_3\right)^{j}\right]&=\mathbb{E}\left[\sum_{m=0}^{j}\binom{j}{m}\left(\frac{k-1}{n-k+2}u_3\right)^m\right]\\
&=\sum_{m=0}^{j}\binom{j}{m}\left(\frac{k-1}{n-k+2}\right)^m\mathbb{E}\left[u_3^m\right].
\end{aligned}
$$

For $m\geq 1$ it holds that (see Walck (1996, Chapter 32.2))

$$
\begin{aligned}
c_m &:= \left(\frac{k-1}{n-k+2}\right)^m \mathbb{E}\left[u_3^m\right] \\
&= \frac{\Gamma\left(\frac{n-k+2}{2}-m\right)}{\Gamma\left(\frac{n-k+2}{2}\right)} \frac{\Gamma\left(\frac{k-1}{2}+m\right)}{\Gamma\left(\frac{k-1}{2}\right)} e^{-\frac{s}{2}} \; {}_1F_1\left(m+\frac{k-1}{2}; \frac{k-1}{2}; \frac{s}{2}\right) \\
&= \frac{(k-1+2(m-1))\dots(k-1)}{(n-k-2(m-1))\dots(n-k)} e^{-\frac{s}{2}} \; {}_1F_1\left(m+\frac{k-1}{2}; \frac{k-1}{2}; \frac{s}{2}\right). \qquad (4)
\end{aligned}
$$

Finally, putting all the terms together we get the statement of the theorem. □

Now we consider an explicit formula for the higher oder central moments of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ which is given in the next corollary.

Corollary 1. *Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$, $n > k$ and $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$ with $\lambda > 0$ and positive definite $\mathbf{\Sigma}$. Furthermore, let $\mathbf{A}$ and $\mathbf{z}$ be independent and $\mathbf{l}$ be a k-dimensional vector of constants. Then the r-th order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by*

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} - \mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}]\right)^r\right] = (-\kappa_1)^r + \sum_{i=1}^r \binom{r}{i} \frac{(-\kappa_1)^{r-i}}{(n-k-1)\dots(n-k-2i+1)} \\
&\times \left[(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^i + \sum_{j=1}^{\lfloor i/2 \rfloor} \binom{i}{2j} \frac{(2j)!}{2^j j!} \left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{i-2j} \left(\lambda \mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^j \left(1 + \sum_{m=1}^{j} \binom{j}{m} c_m\right)\right]
\end{aligned}
$$

for $n-k+1 > 2r$ with $\kappa_1 = \frac{1}{n-k-1}\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$ and

$$
c_m = \frac{(k-1+2(m-1))\dots(k-1)}{(n-k-2(m-1))\dots(n-k)} e^{-\frac{s}{2}} \; {}_1F_1\left(m+\frac{k-1}{2}; \frac{k-1}{2}; \frac{s}{2}\right),
$$

where $s = \boldsymbol{\mu}^T\mathbf{R}_\mathbf{l}\boldsymbol{\mu}/\lambda$ and $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$.

Proof. From Theorem 1 it follows that

$$
\kappa_1 := \mathbb{E}\left[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}\right] = \frac{1}{n-k-1}\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}. \qquad (5)
$$

Using the binomial formula and properties of the mathematical expectation, we obtain that

$$
\begin{aligned}
\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} - \kappa_1\right)^r\right] &= \mathbb{E}\left[\sum_{i=0}^r \binom{r}{i} (-\kappa_1)^{r-i} (\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z})^i\right] \qquad (6) \\
&= (-\kappa_1)^r + \sum_{i=1}^r \binom{r}{i} (-1)^{r-i}\kappa_1^{r-i} \mathbb{E}\left[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z})^i\right]. \qquad (7)
\end{aligned}
$$

Finally, applying Theorem 1 we get the statement of the corollary. $\square$

In the following corollary, we deliver the expressions of the second order central moment, the third order central moment, and the fourth order central moment for $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ in closed-forms without using the confluent hypergeometric function.

Corollary 2. *Let $\mathbf{A} \sim \mathcal{W}_k(n,\mathbf{\Sigma})$, $n > k$ and $\mathbf{z} \sim \mathcal{N}_k(\boldsymbol{\mu}, \lambda\mathbf{\Sigma})$ with $\lambda > 0$ and positive definite $\mathbf{\Sigma}$. Furthermore, let $\mathbf{A}$ and $\mathbf{z}$ be independent and $\mathbf{l}$ be a k-dimensional vector of constants. Also, let $s = \boldsymbol{\mu}^T\mathbf{R}_\mathbf{l}\boldsymbol{\mu}/\lambda$ with $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$ Then*

(a) the second order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by

$$\mathbb{E}[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} - \mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}])^2] = d_1^{(0)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^2 + d_2^{(0)}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l},$$

for $n-k>3$ with

$$d_1^{(0)} = \frac{n-k+1}{(n-k)(n-k-1)^2(n-k-3)} \quad \text{and} \quad d_2^{(0)} = \frac{\lambda(n-1)+\boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}{(n-k)(n-k-1)(n-k-3)};$$

(b) the third order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by

$$\mathbb{E}[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} - \mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}])^3] = d_1^{(1)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^3 + d_2^{(1)}\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\cdot\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$$

for $n-k>5$ with

$$\begin{aligned} d_1^{(1)} &= \frac{16}{(n-k-1)^3(n-k-3)(n-k-5)}, \\ d_2^{(1)} &= \frac{12\lambda}{(n-k-1)^2(n-k-3)(n-k-5)}\left(1+\frac{s+k-1}{n-k}\right); \end{aligned}$$

(c) the fourth order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by

$$\mathbb{E}[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} - \mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}])^4] = d_1^{(3)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^4 + d_2^{(3)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^2\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + d_3^{(3)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l})^2$$

for $n-k>7$ with

$$\begin{aligned} d_1^{(2)} &= \frac{3[(n-k+1)(n-k-5)(n-k-7)-(n-k-1)^2(n-k-9)]}{(n-k-1)^4(n-k-3)(n-k-5)(n-k-7)}, \\ d_2^{(2)} &= \frac{6\lambda(1+c_1)[(n-k-1)^2-(n-k+3)(n-k-7)]}{(n-k-1)^3(n-k-3)(n-k-5)(n-k-7)}, \\ d_3^{(2)} &= \frac{3\lambda^2(1+2c_1+c_2)}{(n-k-1)(n-k-3)(n-k-5)(n-k-7)}, \end{aligned}$$

with

$$c_1 = \frac{s+k-1}{n-k} \qquad \textit{and} \qquad c_2 = \frac{s^2+(2s+k-1)(k+1)}{(n-k)(n-k-2)}.$$

Proof. From Theorem 1 and Walck (1996, Chapter 32.2) we obtain that

$$\begin{aligned} c_1 &= \frac{k-1}{n-k+2}\mathbb{E}[u_3] = \frac{s+k-1}{n-k}, \\ c_2 &= \left(\frac{k-1}{n-k+2}\right)^2 \mathbb{E}[u_3^2] = \frac{s^2+(2s+k-1)(k+1)}{(n-k)(n-k-2)}. \end{aligned}$$

Using Corollary 1, we get that the second order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$ is given by

$$\begin{aligned} &\mathbb{E}[(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z} - \mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}])^2] = (-\kappa_1)^2 + \sum_{i=1}^{2}\binom{2}{i}\frac{(-\kappa_1)^{2-i}}{(n-k-1)...(n-k-2i+1)} \\ &\times \left[(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^i + \sum_{j=1}^{\lfloor i/2 \rfloor}\binom{i}{2j}\frac{(2j)!}{2^j j!}\left(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{i-2j}\left(\lambda\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\right)^j\left(1+\sum_{m=1}^{j}\binom{j}{m}c_m\right)\right] \\ = &\ \kappa_1^2 - \frac{2\kappa_1}{n-k-1}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{(n-k-1)(n-k-3)} \\ &\times \left[(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2 + \lambda\left(1+\frac{s+k-1}{n-k}\right)\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\right] \\ = &\ \left[\frac{1}{(n-k-1)(n-k-3)} - \frac{1}{(n-k-1)^2}\right](\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2 \\ &+\frac{\lambda}{(n-k-1)(n-k-3)}\left(1+\frac{s+k-1}{n-k}\right)\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l} \\ = &\ d_1^{(0)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2 + d_2^{(0)}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l} \end{aligned}$$

with $d_1^{(0)}$ and $d_2^{(0)}$ as defined in the formulation of the corollary.

In order to derive the third order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$, it holds that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}-\mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}]\right)^3\right] = (-\kappa_1)^3 + \sum_{i=1}^{3}\binom{3}{i}\frac{(-\kappa_1)^{3-i}}{(n-k-1)...(n-k-2i+1)} \\
&\times\left[(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^i + \sum_{j=1}^{\lfloor i/2\rfloor}\binom{i}{2j}\frac{(2j)!}{2^j j!}\left(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^{i-2j}\left(\lambda\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^j\left(1+\sum_{m=1}^{j}\binom{j}{m}c_m\right)\right] \\
= \; & -\kappa_1^3 + \frac{3\kappa_1^2}{n-k-1}\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \frac{3\kappa_1}{(n-k-1)(n-k-3)}\left[(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^2+\lambda(1+c_1)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right] \\
&+\frac{1}{(n-k-1)(n-k-3)(n-k-5)}\left[(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^3+3\lambda(1+c_1)\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\cdot\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right] \\
= \; & \left[\frac{2}{(n-k-1)^3}-\frac{3}{(n-k-1)^2(n-k-3)}+\frac{1}{(n-k-1)(n-k-3)(n-k-5)}\right](\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^3 \\
&+3\lambda\left[\frac{1}{(n-k-1)(n-k-3)(n-k-5)}-\frac{1}{(n-k-1)^2(n-k-3)}\right] \\
&\times\left(1+\frac{s+k-1}{n-k}\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\cdot\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} \\
= \; & d_1^{(1)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})^3 + d_2^{(1)}\mathbf{l}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\cdot\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}
\end{aligned}
$$

with $d_1^{(1)}$ and $d_2^{(1)}$ which are defined in the statement of the corollary.

Finally, we derive the fourth order central moment of $\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}$. From Corollary 1, we have

$$\mathbb{E}\left[\left(\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}-\mathbb{E}[\mathbf{l}^T\mathbf{A}^{-1}\mathbf{z}]\right)^4\right]=(-\kappa_1)^4+\sum_{i=1}^{4}\binom{4}{i}\frac{(-\kappa_1)^{4-i}}{(n-k-1)...(n-k-2i+1)}$$

$$\times\left[(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^i+\sum_{j=1}^{\lfloor i/2\rfloor}\binom{i}{2j}\frac{(2j)!}{2^j j!}\left(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^{i-2j}\left(\lambda\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\right)^j\left(1+\sum_{m=1}^{j}\binom{j}{m}c_m\right)\right]$$

$$\begin{aligned}
=\ & \frac{1}{(n-k-1)^4}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^4-\frac{4\kappa_1^3}{n-k-1}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}+\frac{6\kappa_1^2}{(n-k-1)(n-k-3)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2\\
& +\frac{6\lambda(1+c_1)\kappa_1^2}{(n-k-1)(n-k-3)}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}-\frac{4\kappa_1}{(n-k-1)(n-k-3)(n-k-5)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^3\\
& -\frac{12\lambda(1+c_1)\kappa_1}{(n-k-1)(n-k-3)(n-k-5)}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\cdot\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\\
& +\frac{1}{(n-k-1)\dots(n-k-7)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^4+\frac{6\lambda(1+c_1)}{(n-k-1)\dots(n-k-7)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\\
& +\frac{3\lambda^2(1+2c_1+c_2)}{(n-k-1)\dots(n-k-7)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l})^2\\
=\ & \left[\frac{6}{(n-k-1)^3(n-k-3)}-\frac{3}{(n-k-1)^4}-\frac{4}{(n-k-1)^2(n-k-3)(n-k-5)}\right.\\
& \left.+\frac{1}{(n-k-1)\dots(n-k-7)}\right](\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^4\\
& +\left[\frac{6\lambda(1+c_1)}{(n-k-1)^3(n-k-3)}-\frac{12\lambda(1+c_1)}{(n-k-1)^2(n-k-3)(n-k-5)}\right.\\
& \left.+\frac{6\lambda(1+c_1)}{(n-k-1)\dots(n-k-7)}\right](\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\\
& +\frac{3\lambda^2(1+2c_1+c_2)}{(n-k-1)\dots(n-k-7)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l})^2\\
=\ & d_1^{(2)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^4+d_2^{(2)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}+d_3^{(2)}(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l})^2
\end{aligned}$$

with $d_1^{(2)}$, $d_2^{(2)}$, and $d_3^{(2)}$ as defined in the formulation of the corollary. It completes the proof of the corollary. □

3 Main Results

We consider a portfolio that consists of k assets. Let $\mathbf{x}_t=(x_{1t},\dots,x_{kt})^T$ be the k-dimensional vector of log-returns of these assets at time $t=1,\dots,n$. The weight of the ith asset in the portfolio is denoted by w_i and let $\mathbf{w}=(w_1,\cdots,w_k)^T$ be the vector of weights. Let the mean vector of the asset returns be denoted by $\boldsymbol{\mu}$ and the covariance matrix by $\boldsymbol{\Sigma}$ which is assumed to be positive definite. Notice that the mean-variance portfolio problem is equivalent to maximizing the expected quadratic utility. Since the risk is usually measured by the variance of the portfolios return, the optimal portfolio without a risk-free asset is obtained by minimizing the portfolio variance for a given level under the constraint $\mathbf{w}^T\mathbf{1}_k = 1$ where $\mathbf{1}_k$ denotes the vector of ones. However, if short selling is allowed and a risk-free asset, with return r_f, is available, then part of investor's wealth is invested into the risk-free asset, whereas the rest of the wealth is invested into the portfolio from the efficient frontier. The return of risky assets is given as $\mu_p = \mathbf{w}^T(\boldsymbol{\mu} - r_f\mathbf{1}_k) + r_f$ with the variance $\sigma_p^2 = \mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$. Maximizing the utility leads us to the tangency portfolio weights which are given by

$$\mathbf{w}_{TP} = \alpha^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}). \tag{8}$$

where α denotes the investor's attitude towards risk. The tangency portfolio lies on the intersection of the mean-variance frontier and the tangency line drawn from the portfolio consisting of the risk-free asset (Ingersoll (1987)).

Throughout the paper it is assumed that the asset returns $\mathbf{x}_1, \dots, \mathbf{x}_n$ are independent and identically normally distributed, i.e. $\mathbf{x}_t \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $t = 1, \dots, n$. Since $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown parameters the investor cannot determine $\mathbf{w}_{TP}$. In practice, these parameters need to be estimated. Here we use the corresponding sample estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ expressed as

$$\overline{\mathbf{x}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j \quad \text{and} \quad \mathbf{S} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^T.$$

Replacing $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ with $\overline{\mathbf{x}}$ and $\mathbf{S}$ in (8), we obtain the sample estimator $\hat{\mathbf{w}}_{TP}$ of tangency portfolio weights $\mathbf{w}_{TP}$, i.e.

$$\widehat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{S}^{-1}(\overline{\mathbf{x}} - r_f\mathbf{1}).$$

In this paper we focus on the linear combination of the tangency portfolio weights. In particular, we are interested in

$$\theta = \mathbf{l}^T\mathbf{w}_{TP} = \alpha^{-1}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}),$$

where $\mathbf{l}$ is a k-dimensional vector of constants. Then the sample estimator of θ is given by

$$\hat{\theta} = \mathbf{l}^T\widehat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{L}\mathbf{S}^{-1}(\overline{\mathbf{x}} - r_f\mathbf{1}).$$

We now utilize fundamental results obtained in Theorem 1 and Corollary 2 to derive explicit expressions for the higher order non-central and central moments of $\hat{\theta} = \mathbf{l}^T\widehat{\mathbf{w}}_{TP}$.

Theorem 2. *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be identically and independently distributed random vectors with* $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $k < n-1$ *and* $\boldsymbol{\Sigma} > 0$. *Also, let* $\mathbf{l}$ *be a* k*-dimensional vector of constants,* $\breve{\mathbf{l}} = (n-1)/\alpha\mathbf{l}$, $\breve{\boldsymbol{\mu}} = \boldsymbol{\mu} - r_f\mathbf{1}_k$, *and* $\breve{s} = n\breve{\boldsymbol{\mu}}^T\mathbf{R}_{\breve{\mathbf{l}}}\breve{\mu}$ *with* $\mathbf{R}_{\breve{\mathbf{l}}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\breve{\mathbf{l}}\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}/\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\mathbf{l}}$. *Then*

(a) the r*-th order moment of* $\hat{\theta}$ *is given by*

$$\begin{aligned}\mu_r := \mathbb{E}[\hat{\theta}^r] &= \frac{1}{(n-k-2)\ldots(n-k-2r)} \\ &\times \left[(\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}})^r + \sum_{j=1}^{\lfloor r/2\rfloor}\binom{r}{2j}\frac{(2j)!}{(2n)^j j!}\left(\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}\right)^{r-2j}\left(\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\mathbf{l}}\right)^j\right. \\ &\times \left.\left(1+\sum_{m=1}^{j}\binom{j}{m}\left(\frac{k-1}{n-k+1}\right)^m\breve{c}_m\right)\right], \qquad n-k>2r\end{aligned}$$

where

$$\breve{c}_m = \frac{(k-1+2(m-1))\ldots(k-1)}{(n-k-2(m-1)-1)\ldots(n-k-1)}e^{-\frac{\breve{s}}{2}}\,{}_1\widetilde{F}_1\left(m+\frac{k-1}{2};\frac{k-1}{2};\frac{\breve{s}}{2}\right).$$

(b) the r*-th order central moment of* $\hat{\theta}$ *is given by*

$$\begin{aligned}\overline{\mu}_r := \mathbb{E}[(\hat{\theta}-\mu_1)^r] &= (-\mu_1)^r + \sum_{i=1}^{r}\binom{r}{i}\frac{(-\mu_1)^{r-i}}{(n-k-2)\ldots(n-k-2i)} \\ &\times \left[(\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}})^i + \sum_{j=1}^{\lfloor i/2\rfloor}\binom{i}{2j}\frac{(2j)!}{(2n)^j j!}\left(\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\boldsymbol{\mu}}\right)^{i-2j}\left(\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\mathbf{l}}\right)^j\right. \\ &\times \left.\left(1+\sum_{m=1}^{j}\binom{j}{m}\left(\frac{k-1}{n-k+1}\right)^m\breve{c}_m\right)\right], \qquad n-k>2r.\end{aligned}$$

Proof. From Muirhead (1982, Chapter 3) we have that

$$\begin{aligned}\overline{\mathbf{x}} &\sim \mathcal{N}_k\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right), \\ \mathbf{V} := (n-1)\mathbf{S} &\sim \mathcal{W}_k(n-1, \boldsymbol{\Sigma}),\end{aligned}$$

moreover, $\overline{\mathbf{x}}$ and $\mathbf{V}$ are independently distributed. Since

$$\hat{\theta} = \mathbf{l}^T\hat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{l}^T\mathbf{S}^{-1}(\overline{\mathbf{x}} - r_f\mathbf{1}_k) = \frac{n-1}{\alpha}\mathbf{l}^T\mathbf{V}^{-1}(\overline{\mathbf{x}} - r_f\mathbf{1}_k).$$

the rest of the proof follows from Theorem 1 and Corollary 2. □

The following corollary delivers the expressions of the mean and the variance for $\widehat{\mathbf{w}}_{TP}$.

Corollary 3. *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be identically and independently distributed random vectors with* $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$*,* $k < n-1$ *and* $\boldsymbol{\Sigma} > 0$*. Also let* $\breve{\boldsymbol{\mu}} = \boldsymbol{\mu} - r_f \mathbf{1}_k$ *and* $\delta = n\breve{\boldsymbol{\mu}}^T \boldsymbol{\Sigma}^{-1} \breve{\boldsymbol{\mu}}$*. Then the mean and the variance of* $\widehat{\mathbf{w}}_{TP}$ *are given by*

$$\mathbb{E}[\hat{\mathbf{w}}_{TP}] = \frac{n-1}{n-k-2}\mathbf{w}_{TP}$$

and

$$\mathbb{V}ar[\hat{\mathbf{w}}_{TP}] = \breve{d}_1^{(0)} \mathbf{w}_{TP}\mathbf{w}_{TP}^T + \breve{d}_2^{(0)} \boldsymbol{\Sigma}^{-1}$$

with

$$\breve{d}_1^{(0)} = \frac{(n-k)(n-1)^2}{(n-k-1)(n-k-2)^2(n-k-4)} \quad \text{and} \quad \breve{d}_2^{(0)} = \frac{(n-1)^2(n-2+\delta)}{n\alpha^2(n-k-1)(n-k-2)(n-k-4)}.$$

Proof. From Corollary 2 we get the first two moments of $\hat{\theta}$ which are given by

$$\mathbb{E}[\hat{\theta}] = \frac{n-1}{n-k-2}\theta$$

and

$$\mathbb{V}ar[\hat{\theta}] = \breve{d}_1^{(0)}\theta^2 + \breve{d}_2^{(0)}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}$$

with

$$\breve{d}_1^{(0)} = \frac{(n-k)(n-1)^2}{(n-k-1)(n-k-2)^2(n-k-4)} \quad \text{and} \quad \breve{d}_2^{(0)} = \frac{(n-1)^2(n-2+\delta)}{n\alpha^2(n-k-1)(n-k-2)(n-k-4)}.$$

Moreover, since $\mathbf{l}$ is an arbitrary vector of constants we get the statement of the corollary. □

In the next corollary, we derive the expressions for skewness and the kurtosis of $\hat{\theta} = \mathbf{l}^T\widehat{\mathbf{w}}_{TP}$.

Corollary 4. *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *be identically and independently distributed random vectors with* $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$*,* $k < n-1$ *and* $\boldsymbol{\Sigma} > 0$*. Also, let* $\mathbf{l}$ *be a* k*-dimensional vector of constants,* $\breve{\mathbf{l}} = (n-1)/\alpha\mathbf{l}$*,* $\breve{\boldsymbol{\mu}} = \boldsymbol{\mu} - r_f\mathbf{1}_k$*, and* $\breve{s} = n\breve{\boldsymbol{\mu}}^T\mathbf{R}_{\breve{\mathbf{l}}}\breve{\mu}$ *with* $\mathbf{R}_{\breve{\mathbf{l}}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\breve{\mathbf{l}}\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}/\breve{\mathbf{l}}^T\boldsymbol{\Sigma}^{-1}\breve{\mathbf{l}}$*. Then the skewness* $\hat{\theta}$ *is given by*

$$\text{Skewness}[\hat{\theta}] \;=\; \left(\breve{d}_1^{(1)}\theta^3 + \breve{d}_2^{(1)}\theta\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\right)\left(\breve{d}_1^{(0)}\theta^2 + \breve{d}_2^{(0)}\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}\right)^{-3/2}$$

with $\breve{d}_1^{(0)}$ *and* $\breve{d}_2^{(0)}$ *which are defined in Corollary 3, and*

$$\begin{aligned}
\breve{d}_1^{(1)} &= \frac{16(n-1)^3}{(n-k-2)^3(n-k-4)(n-k-6)}, \\
\breve{d}_2^{(1)} &= \frac{12(n-1)^3}{\alpha^2 n(n-k-2)^2(n-k-4)(n-k-6)}\left(1+\frac{\breve{s}+k-1}{n-k-1}\right),
\end{aligned}$$

while the kurtosis of $\hat{\theta}$ *is expressed as*

$$\text{Kurtosis}[\hat{\theta}] = \left(\breve{d}_1^{(3)}\theta^4 + \breve{d}_2^{(3)}\theta^2\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + \breve{d}_3^{(3)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l})^2\right)\left(\breve{d}_1^{(0)}\theta^2 + \breve{d}_2^{(0)}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{-2}.$$

where

$$\begin{aligned}
\breve{d}_1^{(3)} &= \frac{3(n-1)^4[(n-k)(n-k-6)(n-k-8)-(n-k-2)^2(n-k-10)]}{(n-k-2)^4(n-k-4)(n-k-6)(n-k-8)}, \\
\breve{d}_2^{(3)} &= \frac{6(1+\breve{c}_1)(n-1)^4[(n-k-2)^2-(n-k+2)(n-k-8)]}{\alpha^2 n(n-k-2)^3(n-k-4)(n-k-6)(n-k-8)}, \\
\breve{d}_3^{(3)} &= \frac{3(1+2\breve{c}_1+\breve{c}_2)(n-1)^4}{\alpha^4 n^2(n-k-2)(n-k-4)(n-k-6)(n-k-8)},
\end{aligned}$$

with

$$\breve{c}_1 = \frac{\breve{s}+k-1}{n-k-1} \quad \textit{and} \quad \breve{c}_2 = \frac{\breve{s}^2+(2\breve{s}+k-1)(k+1)}{(n-k-1)(n-k-3)}.$$

Proof. The skewness of $\hat{\theta}$ is given by

$$\text{Skewness}[\hat{\theta}] = \frac{\overline{\mu}_3}{\left[\mathbb{V}ar(\hat{\theta})\right]^{3/2}} = \overline{\mu}_3\left(\breve{d}_1^{(0)}\theta^2 + \breve{d}_2^{(0)}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{-3/2},$$

where $\mathbb{V}ar(\hat{\theta})$ is obtained from Corollary 4. From Corollary 2 it follows that

$$\overline{\mu}_3 = \tilde{d}_1^{(1)}(\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\boldsymbol{\mu}})^3 + \tilde{d}_2^{(1)}\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\boldsymbol{\mu}}\cdot\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\mathbf{l}},$$

where

$$\begin{aligned}
\tilde{d}_1^{(1)} &= \frac{16}{(n-k-2)^3(n-k-4)(n-k-6)} \\
\tilde{d}_2^{(1)} &= \frac{12}{n(n-k-2)^2(n-k-4)(n-k-6)}\left(1+\frac{\breve{s}+k-1}{n-k-1}\right)
\end{aligned}$$

with $\breve{s} = n\breve{\boldsymbol{\mu}}^T\mathbf{R}_{\breve{\mathbf{l}}}\breve{\boldsymbol{\mu}}$ and $\mathbf{R}_{\breve{\mathbf{l}}} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\breve{\mathbf{l}}\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}/\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\mathbf{l}}$. Furthermore, $\overline{\mu}_3$ can be rewritten in the next form

$$
\begin{aligned}
\overline{\mu}_3 &= \tilde{d}_1^{(1)}(n-1)^3\theta^3 + \tilde{d}_2^{(1)}\alpha^{-2}(n-1)^3\theta\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} \\
&= \breve{d}_1^{(1)}\theta^3 + \breve{d}_2^{(1)}\theta\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l},
\end{aligned}
$$

where $\breve{d}_1^{(1)}$ and $\breve{d}_2^{(1)}$ are the same as in the formulation of the corollary. Putting all above together we get the skewness of $\hat{\theta}$.
We later move on and derive the explicit formula for the kurtosis of $\hat{\theta}$. It holds that

$$
\text{Kurtosis}[\hat{\theta}] = \frac{\overline{\mu}_4}{\left[\mathbb{V}ar(\hat{\theta})\right]^2} = \overline{\mu}_4\left(\breve{d}_1^{(0)}\theta^2 + \breve{d}_2^{(0)}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\right)^{-2}.
$$

Using Corollary 2 we get

$$
\overline{\mu}_4 = \tilde{d}_1^{(3)}(\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\boldsymbol{\mu}})^4 + \tilde{d}_2^{(3)}(\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\boldsymbol{\mu}})^2\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\mathbf{l}} + \tilde{d}_3^{(3)}(\breve{\mathbf{l}}^T\mathbf{\Sigma}^{-1}\breve{\mathbf{l}})^2
$$

where

$$
\begin{aligned}
\tilde{d}_1^{(3)} &= \frac{3[(n-k)(n-k-6)(n-k-8)-(n-k-2)^2(n-k-10)]}{(n-k-2)^4(n-k-4)(n-k-6)(n-k-8)}, \\
\tilde{d}_2^{(3)} &= \frac{6(1+\breve{c}_1)[(n-k-2)^2-(n-k+2)(n-k-8)]}{n(n-k-2)^3(n-k-4)(n-k-6)(n-k-8)}, \\
\tilde{d}_3^{(3)} &= \frac{3(1+2\breve{c}_1+\breve{c}_2)}{n^2(n-k-2)(n-k-4)(n-k-6)(n-k-8)},
\end{aligned}
$$

with

$$
\breve{c}_1 = \frac{\breve{s}+k-1}{n-k-1} \quad \text{and} \quad \breve{c}_2 = \frac{\breve{s}^2+(2\breve{s}+k-1)(k+1)}{(n-k-1)(n-k-3)}.
$$

Moreover, $\overline{\mu}_4$ can be rewritten as

$$
\overline{\mu}_4 = \breve{d}_1^{(3)}\theta^4 + \breve{d}_2^{(3)}\theta^2\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + \breve{d}_3^{(3)}(\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l})^2
$$

where $\breve{d}_1^{(3)}$, $\breve{d}_2^{(3)}$, and $\breve{d}_3^{(3)}$ are the same as in the formulation of the corollary. It completes the proof of the corollary. □

4 Empirical Illustration

In this section we present the results of empirical study where we show how theoretical results of the previous section can be applied to real data. We consider weekly data of $k = 8$ financial indexes which are listed in NASDAQ stock exchange. Their abbreviated symbolic names are `RCMP, IXTS, IXCO, TRAN, INDS, NBI, IXIC, BANK`. The data are taken for the period from August 2007 to April 2017. Weekly log returns on each index have been considered. The weekly log returns on the three-month US treasury bill are used as the risk-free rate. The risk aversion coefficient α is taken as 50.

Table 1 presents the mean, variance, skewness, and kurtosis for the densities of the estimated tangency portfolio weights. The sample size n is set to be $\{50, 100, 250, 500\}$. We observe that some expected values are negative, which means that we have a short sales for corresponding financial index. The variance of weights decreases significantly with the increase in sample size n. Moreover, when n increases the skewness tends to approach 0, while the kurtosis remains closer to 3 in all cases, thereby, indicating that the distribution of estimated weights satisfies the normality assumption. Through this empirical exercise, we confirm that with a larger sample size the distribution of the tangency portfolio weights can be nicely approximated by the normal distribution.

	RCMP	IXTS	IXCO	TRAN	INDS	NBI	IXIC	BANK
n=50								
Mean	0.345032	0.366515	4.040067	-0.088892	3.133680	0.791789	-9.273100	0.765419
Variance	0.181189	0.418301	10.845010	0.136999	9.958039	0.724571	69.775950	0.279168
Skewness	0.265720	0.187369	0.393428	-0.079937	0.322791	0.303303	-0.358553	0.457357
Kurtosis	3.673955	3.636377	3.766017	3.606461	3.710161	3.696932	3.736855	3.828065
n=100								
Mean	-0.162695	0.004987	1.735233	0.006186	1.288362	0.321607	-3.708634	0.481352
Variance	0.049531	0.082501	2.881014	0.037299	2.505783	0.220435	18.010550	0.066849
Skewness	-0.101411	0.002422	0.141026	0.004469	0.112747	0.095092	-0.120923	0.249975
Kurtosis	3.271306	3.261263	3.280801	3.261277	3.273696	3.270086	3.275581	3.324378
n=250								
Mean	-0.317376	-0.015590	1.122693	0.037473	1.027947	0.404376	-2.576696	0.313285
Variance	0.017011	0.025321	0.770156	0.011379	0.580828	0.058658	4.492414	0.023227
Skewness	-0.120780	-0.004982	0.064620	0.017854	0.068081	0.083944	-0.061447	0.102745
Kurtosis	3.113364	3.099390	3.103303	3.099666	3.103740	3.106041	3.102924	3.109425
n=500								
Mean	-0.276951	-0.050477	0.848870	0.002643	0.774054	0.303571	-1.723870	0.071699
Variance	0.006687	0.013159	0.204863	0.004343	0.132618	0.016668	1.185713	0.007898
Skewness	-0.081740	-0.010861	0.0459880	0.000990	0.052017	0.057427	-0.038897	0.019895
Kurtosis	3.055911	3.049724	3.051577	3.049616	3.052130	3.052686	3.051016	3.049980

Table 1: Mean, variance, skewness and kurtosis of the estimator for the eight financial indexes from NASDAQ stock exchange.

5 Conclusions

In this paper we study higher order moments of the estimated tangency portfolio weights obtained under the assumption of normally and independently distributed returns. In particular, we derive the higher order non-central and central moments of estimated weights that depend on the confluent hypergeometric function. Moreover, we provide the expressions of the mean, variance, skewness and kurtosis in a closed-forms without using the confluent hypergeometric function. The results are supported by an empirical study, where such expressions are evaluated for the actual returns for eight financial indexes listed in NASDAQ stock exchange.

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