Skew-Symmetric Matrix Pencils: Stratification Theory and Tools

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Licentiate Thesis

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Abstract

Investigating the properties, explaining, and predicting the behaviour of a physical system described by a system (matrix) pencil often require the understanding of how canonical structure information of the system pencil may change, e.g., how eigenvalues coalesce or split apart, due to perturbations in the matrix pencil elements. Often these system pencils have different block-partitioning and / or symmetries.

We study changes of the congruence canonical form of a complex skew-symmetric matrix pencil under small perturbations. The problem of computing the congruence canonical form is known to be ill-posed: both the canonical form and the reduction transformation depend discontinuously on the entries of a pencil. Thus it is important to know the canonical forms of all such pencils that are close to the investigated pencil. One way to investigate this problem is to construct the stratification of orbits and bundles of the pencils. To be precise, for any problem dimension we construct the closure hierarchy graph for congruence orbits or bundles. Each node (vertex) of the graph represents an orbit (or a bundle) and each edge represents the cover/closure relation. Such a relation means that there is a path from one node to another node if and only if a skew-symmetric matrix pencil corresponding to the first node can be transformed by an arbitrarily small perturbation to a skew-symmetric matrix pencil corresponding to the second node. From the graph it is straightforward to identify more degenerate and more generic nearby canonical structures.

A necessary (but not sufficient) condition for one orbit being in the closure of another is that the first orbit has larger codimension than the second one. Therefore we compute the codimensions of the congruence orbits (or bundles). It is done via the solutions of an associated homogeneous system of matrix equations.

The complete stratification is done by proving the relation between equivalence and congruence for the skew-symmetric matrix pencils. This relation allows us to use the known result about the stratifications of general matrix pencils (under strict equivalence) in order to stratify skew-symmetric matrix pencils under congruence.

Matlab functions to work with skew-symmetric matrix pencils and a number of other types of symmetries for matrices and matrix pencils are developed and included in the Matrix Canonical Structure (MCS) Toolbox.

Preface

This Licentiate Thesis consists of the following papers:

Paper I — A. Dmytryshyn, B. Kågström, and V. V. Sergeichuk. Skew-symmetric matrix pencils: Codimension counts and the solution of a pair of matrix equations[1]. *Linear Algebra Appl.*, 438(8) (2013) 3375–3396.

Paper II — A. Dmytryshyn, S. Johansson, and B. Kågström. Codimension computations of congruence orbits of matrices, skew-symmetric and symmetric matrix pencils using Matlab. *Report UMINF* 13.18, Dept. of Computing Science, Umeå University, Sweden, 2013.

Paper III — A. Dmytryshyn and B. Kågström. Orbit closure hierarchies of skew-symmetric matrix pencils. *Report UMINF* 14.02, Dept. of Computing Science, Umeå University, Sweden, 2014.

In addition to the papers included in the thesis, the following publications were written within the studies:

A.R. Dmytryshyn, V. Futorny, and V.V. Sergeichuk. Miniversal deformations of matrices of bilinear forms. *Linear Algebra Appl.*, 436(7) (2012) 2670–2700.

A. Dmytryshyn, B. Kågström, and V.V. Sergeichuk. Symmetric matrix pencils: Codimension counts and the solution of a pair of matrix equations. *Electron. J. Linear Algebra*, (accepted 2014).

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Contents

Chapter 1

Introduction

Changes of canonical structure information, e.g., confluence and splitting of eigenvalues, are essential issues for understanding the properties as well as explaining and predicting the behaviour of a physical system described by a system (matrix) pencil. In general, these problems are known to be ill-posed, i.e., small perturbations in the input parameters may lead to big changes in the answers. Therefore when the matrices (of the system pencil) arise as a result of measures, i.e., their entries are known with errors, the issues about changes of canonical structure information become particularly important [36]. Problems related to these challenges have been in the field of interests of many researchers for several years. One approach is to construct a *closure hierarchy graph* (*stratification*) for orbits or bundles of system (matrix) pencils. Each node (vertex) of the graph represents an orbit (or a bundle) and each edge represents a cover/closure relation, i.e., there is an upwards path from a node $\mathcal{S}_1$ to a node $\mathcal{S}_2$ if and only if $\mathcal{S}_1$ can be transformed by an arbitrarily small perturbation to a system (matrix) pencil whose canonical structure information corresponds to $\mathcal{S}_2$. From an orbit (or bundle) stratification, it is straightforward to identify more degenerate and more generic nearby canonical structures to a given matrix pencil $A - \lambda B$. Such information can give new insight to the model of the underlying physical problem. Going downwards in the closure hierarchy graph, i.e., moving to more degenerate matrix pencils, can be achieved by perturbations of finite but not arbitrarily small sizes.

The stratification theory and algorithms for the computation of the associated closure hierarchy graph(s) are known for matrix pencils under strict equivalence transformations [23, 24, 27], non-singular controllability and observability pairs [26], and full normal rank matrix polynomials [34]. The stratifications of 2×2 and 3×3 matrices of bilinear forms are considered in [28].

For computing and visualization of the closure hierarchy graphs, the Stratigraph tool [32] has been developed. Furthermore, the Matrix Canonical Structure (MCS) Toolbox for Matlab[1] has been evolved [19, 32] to compute and

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handle canonical structure information associated with system (matrix) pencils.

This Licentiate Thesis is focused on developing theory and tools for analyzing skew-symmetric matrix pencils, i.e., $A - \lambda B$, where $A, B \in \mathbb{C}^{n\times n}$, $A = -A^T$ and $B = -B^T$. Skew-symmetric matrix pencils appear in several applications, e.g., multisymplectic partial differential equations [7] and in the design of a passive velocity field controller [35]. Properties of skew-symmetric matrix pencils have been studied, such as canonical forms [40, 41] and pseudospectra [1].

Consider a skew-symmetric $n \times n$ matrix pencil $A - \lambda B$. The equivalence transformation

$$A - \lambda B \mapsto P^{-1}(A - \lambda B)Q, \quad P, Q \in GL_n(\mathbb{C}),$$

where $GL_n(\mathbb{C})$ is a group of all nonsingular $n \times n$ matrices, does not preserve skew-symmetry, i.e., the transformed matrix pencil $P^{-1}(A - \lambda B)Q$ may not be skew-symmetric. Therefore, skew-symmetric matrix pencils should be considered under the structure preserving congruence transformation

$$A - \lambda B \mapsto C^T(A - \lambda B)C, \quad \text{where } C \in GL_n(\mathbb{C}).$$

In this Licentiate Thesis, we explain how the congruence canonical structure information of skew-symmetric matrix pencils may change under perturbations, i.e., we solve the stratification problem. Our solution requires the following main steps:

1. Determine the orbits of skew-symmetric matrix pencils under congruence (i.e., all pencils with the same congruence invariants) and associated canonical forms; for skew-symmetric matrix pencils see [40, 41]. As for general matrix pencils in [24], bundles of skew-symmetric matrix pencils are defined as unions of orbits; see Section 1.1 or Paper III for more details.

2. Compute the codimensions of the orbits and bundles from the structural information of skew-symmetric matrix pencils; see Papers I and II as well as [15].

3. Determine the necessary and sufficient conditions that one congruence orbit of a skew-symmetric matrix pencil is contained in the closure of another; see Paper III.

Note that Papers I, II, and III are the papers [21], [19], and [20], respectively, in the reference list.

All matrices that we consider are over the field of complex numbers.

1.1 Codimension computations

The set of skew-symmetric $n \times n$ matrix pencils congruent to $A - \lambda B$ forms a manifold in the complex $n^2 - n$ dimensional space (both A and B have $n(n-1)/2$

independent parameters). This manifold is the orbit of $A - \lambda B$ under the action of congruence

$$\mathrm{O}^c_{A-\lambda B} = \{C^T(A-\lambda B)C : C \in GL_n(\mathbb{C})\}. \tag{1.1}$$

The vector space

$$\mathrm{T}_{A-\lambda B} \equiv \{(X^T A + AX) - \lambda(X^T B + BX) : X \in \mathbb{C}^{n\times n}\} \tag{1.2}$$

is the tangent space to the congruence orbit of $A - \lambda B$ at the point $A - \lambda B$. The orthogonal complement to $\mathrm{T}_{A-\lambda B}$, with respect to the Frobenius inner product

$$\langle A - \lambda B, C - \lambda D\rangle = \mathrm{trace}(AC^* + BD^*),$$

is called the normal space (denoted by $\mathrm{N}_{A-\lambda B}$) to the congruence orbit. Figure 1 illustrates the geometry of the spaces.

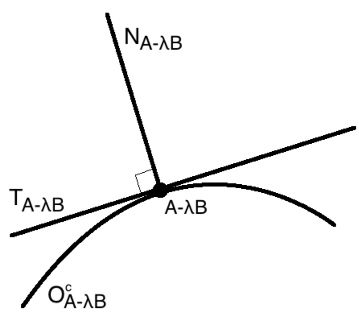

Figure 1: The tangent space $\mathrm{T}_{A-\lambda B}$ and the normal space $\mathrm{N}_{A-\lambda B}$ to the congruence orbit $\mathrm{O}^c_{A-\lambda B}$ at the point $A - \lambda B$.

The *dimension of the orbit* of $A - \lambda B$ is the dimension of its tangent space at the point $A - \lambda B$. The *codimension of the orbit* $A - \lambda B$ is the dimension of the normal space of its orbit at the point $A - \lambda B$, which is equal to $n^2 - n$ minus the dimension of the orbit.

Since the orbit (bundle) of each matrix pencil has only orbits (bundles) with lower codimensions in its closure, codimensions provide a coarse stratification.

In the orbit stratification, the eigenvalues are kept fixed so confluence and splitting of eigenvalues are not allowed (nevertheless, for matrix pencils eigenvalues may appear or disappear). Therefore we also consider stratification of bundles. Let us illustrate with the Jordan canonical form (JCF) of 3×3 matrices. Then an arbitrarily small neighbourhood of $J_3(0)$ (a 3×3 Jordan block corresponding to zero eigenvalue) always contains a matrix with the JCF $J_1(\varepsilon_1) \oplus J_1(\varepsilon_2) \oplus J_1(\varepsilon_3)$ with some (small and different) $\varepsilon_1, \varepsilon_2$, and ε_3. This possible change of the canonical structure information appears in the bundle stratification of 3×3 JCF (an edge from $J_3(0)$ to $J_1(\varepsilon_1) \oplus J_1(\varepsilon_2) \oplus J_1(\varepsilon_3)$ in the graph) but not in the orbit stratification.

As in the case of matrix pencils under strict equivalence [23, 24], two skew-symmetric matrix pencils are in the same bundle $\mathrm{B}^c_{A-\lambda B}$ if and only if they have the same singular structure and the same Jordan structure except that the distinct eigenvalues may be different. Note that a bundle is a union of orbits. For each skew-symmetric matrix pencil $A - \lambda B$ we define

$$\operatorname{cod} \mathrm{B}^c_{A-\lambda B} = \operatorname{cod} \mathrm{O}^c_{A-\lambda B} - \#\{\text{distinct eigenvalues}\}. \tag{1.3}$$

To explain the reason for computing codimensions rather than dimensions, let us refer to the bundle codimensions of Jordan canonical forms in the singularity theory [3, 5]. For bundles of matrices under similarity (i.e., bundles for JCF) the codimension formula (1.3) remains true. Thus distinct eigenvalues that correspond to 1×1 Jordan blocks do not contribute to the bundle codimension. Therefore the codimensions of singularities are independent of the matrix dimensions, e.g., the bundle of $J_3(\mu_1)$ has the same codimension as the bundles $J_3(\mu_1) \oplus J_1(\mu_2)$, $J_3(\mu_1) \oplus J_1(\mu_2) \oplus J_1(\mu_3)$, etc. This property remains true for regular matrix pencils under strict equivalence and regular skew-symmetric matrix pencils under congruence but in both cases it does not hold for singular matrix pencils.

Paper I presents how to compute the codimensions of the congruence orbits of skew-symmetric matrix pencils via the solution of the associated pair of matrix equations. An alternative way to compute the codimensions is to calculate the number of independent parameters in the corresponding miniversal deformations [15] (see also Section 1.1.2 for the definition). Since the matrices are partitioned into blocks according to the canonical forms of the skew-symmetric matrix pencils [41], the diagonal blocks, the off-diagonal blocks that correspond to the canonical summands of the same type, and the off-diagonal blocks that correspond to the canonical summands of different types can be treated independently.

1.1.1 The solution of a pair of matrix equations

Consider a system of homogeneous matrix equations, associated with the matrix representation of the tangent space (1.2) to congruence orbit of $A - \lambda B$ at the point $A - \lambda B$,

$$\begin{aligned} X^T A + AX &= 0, \\ X^T B + BX &= 0, \end{aligned} \tag{1.4}$$

where $A = -A^T$ and $B = -B^T$ are skew-symmetric $n \times n$ matrices. As it is shown in Paper I, the number of linearly independent solutions of (1.4) minus n is equal to the codimension of the congruence orbit of the skew-symmetric matrix pencil $A - \lambda B$.

Without loss of generality, we may consider systems (1.4) in which the skew-symmetric pencil $A - \lambda B$ is in a canonical form under congruence. We use the canonical forms from [41] that are "skew-symmetrized" analogies of the Kronecker canonical forms for matrix pencils under strict equivalence [29].

Beside computing the codimensions in Paper I, we also derive the general solution of (1.4). Recently in a similar way for a square matrix A, the general solution of the matrix equations $XA + AX^\star = 0$ [9, 11, 12, 31], where $\star$ stays for the transposition (T) or conjugate transposition (*), and for matrices A and B of the corresponding sizes, the general solution of the matrix equations $AX + X^\star B = 0$ [13] and $AX + BX^\star = 0$ [10] were derived, as well as for the system (1.4) with both A and B symmetric [22]. The matrix equation $XAX = B$, where A and B are both symmetric or skew symmetric, is studied in [37].

1.1.2 Miniversal deformations of skew-symmetric matrix pencils

We recall that the problem we investigate comes from the fact that reductions to Jordan and Kronecker canonical forms are unstable operations: both the corresponding canonical forms and the reduction transformations depend discontinuously on the elements of the original matrix or matrix pencil. Therefore versal deformations [3] were introduced, i.e., a normal form to which not only a given matrix A (or matrix pencil $A - \lambda B$), but an arbitrary family of matrices $\widetilde{A}$ (or matrix pencils $\widetilde{A} - \lambda\widetilde{B}$) close to it can be reduced by transformations smoothly depending on the elements of $\widetilde{A}$ (or $\widetilde{A} - \lambda\widetilde{B}$). If such a form has the minimal number of independent parameters (as mentioned earlier, this number is equal to the orbit codimension) it is called miniversal deformation. Versal deformations help us to understand which canonical forms we may have in a neighbourhood of a matrix or pencil, i.e., to find the stratification of orbits.

The foundations of this theory were laid by V.I. Arnold (e.g., see [3, 4, 5]). Now miniversal deformations are known for Jordan matrices [3, 23], matrices with respect to congruence [17] and *congruence [18], matrix pencils [23, 30], etc., (a more detailed list of references is given in the introduction of [17]). In particular, miniversal deformations of skew-symmetric and symmetric matrix pencils are derived in [15] and [16], respectively.

1.2 Matrix Canonical Structure Toolbox

Matrix Canonical Structure (MCS) Toolbox [32] for Matlab was developed to work with matrices or matrix pencils under different transformations, e.g., similarity, congruence, equivalence, etc., and the corresponding canonical structures. It is possible to transfer data from MCS Toolbox to StratiGraph and vice versa.

MCS Toolbox includes functions for matrices up to similarity, matrix pencils up to strict equivalence, controllability and observability pairs up to feedback equivalence. The theoretical backgrounds and motivations for these problems are presented in [14, 23, 24, 26]. In Paper II, we extend MCS Toolbox with functionality for congruence and *congruence of matrices, as well as congruence of symmetric and skew-symmetric matrix pencils. Examples include functions that create canonical structure objects or (random) matrix example setups with a desired canonical information, Matlab functions that compute the codimensions of the corresponding orbits, as well as a number of auxiliary functions.

Whenever the canonical structure information of the matrices or the matrix pencils is known (or specified) we use the associated structural information for the codimension computations. Obviously, this computation is always exact and fast for problems of any sizes. Explicit formulas for computing codimensions via canonical structure information for skew-symmetric matrix pencils are derived in Paper I (and for some other cases in [11, 12, 14, 22]). Otherwise, the codimensions are determined numerically by computing the rank and nullity of Kronecker product matrices associated with the problems. The $2n^2 \times n^2$ matrix Z (1.5) is a matrix representation of the tangent space to the congruence orbit of skew-symmetric $n \times n$ matrix pencil $A - \lambda B$ at the point $A - \lambda B$:

$$Z \equiv \begin{bmatrix} A^T \otimes I_n + (I_n \otimes A)P \\ B^T \otimes I_n + (I_n \otimes B)P \end{bmatrix}, \tag{1.5}$$

where P is the $n^2 \times n^2$ permutation matrix that can "transpose" $n \times n$ matrices, i.e., $\mathrm{vec}(X^T) = P\,\mathrm{vec}(X)$ for any $n \times n$ matrix X. The nullities of (1.5) minus n is equal to the codimensions of the congruence orbits of skew-symmetric matrix pencils.

1.3 Orbit closure hierarchies of skew-symmetric matrix pencils

In Paper III, we study how the congruence canonical form (canonical structure information) of a complex skew-symmetric matrix pencil $A - \lambda B$ changes under small perturbations via constructing the orbit and bundle stratifications, respectively. To be precise, for any problem dimension n we construct the closure hierarchy graph for congruence orbits or bundles. Also here, each node (vertex) of the graph represents an orbit (or a bundle) and each edge represents the cover/closure relation, i.e., there is a path from a node $A - \lambda B$ to a node $C - \lambda D$ if and only if $A - \lambda B$ can be transformed by an arbitrarily small perturbation to a skew-symmetric matrix pencil whose canonical form is the one of $C - \lambda D$. As a result, we get qualitative information about the nearby matrix pencils and associated canonical forms. For example, let us consider the closure hierarchy graph for congruence orbits of skew-symmetric 3×3 matrix pencils (only three nodes with codimensions 0, 3, and 6):

$$
\begin{array}{cc}
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{codim. } 0 \\
\uparrow & \\
\begin{bmatrix} 0 & \mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{codim. } 3 \\
\uparrow & \\
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{codim. } 6
\end{array}
\tag{1.6}
$$

From the graph (1.6) it follows that an arbitrarily small neighbourhood of a matrix pencil with the skew-symmetric canonical form

$$
\begin{bmatrix} 0 & \mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

contains a matrix pencil with the canonical form

$$
\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

For a more realistic example, we refer to the closure hierarchy graph for congruence orbits of skew-symmetric 4×4 matrix pencils presented in Paper III (see also the cover of this thesis).

In Paper III, we generalize the fact that two skew-symmetric matrix pencils are equivalent if and only if they are congruent, e.g., see [29, Theorem 6, p.41] or [39, Theorem 3, p.275]. This allows us to reduce the problem of stratification of skew-symmetric matrix pencils to the problem of stratification of matrix pencils under strict equivalence [24]. We also present the feasible stratification method for skew-symmetric matrix pencils.

1.4 Future work

In the recent paper [38] skew-symmetric matrix polynomials are investigated, in particular, a template for the linearization of the skew-symmetric polynomials of odd degrees or regular skew-symmetric polynomials of any degrees is presented. These results together with the stratification of skew-symmetric matrix pencils

(Paper III) form a background to the stratification of linearizations of skew-symmetric matrix polynomials.

We are also interested in matrix pencils with other types of symmetries. In particular, some preparatory work for symmetric matrix pencils has been done in [16, 22]. For pencils with mixed symmetry properties we refer to [11, 17] (one symmetric and one skew-symmetric matrix) and [12, 18] (one hermitian and one skew-hermitian matrix).

Many matrix pencils coming from applications have block structures, e.g., system pencils describing descriptor or state-space systems, controllability and observability pairs. To investigate changes of their invariants is also a part of our future work. The stratifications of controllability and observability pairs have been investigated in [26].

An open problem about the distance to more degenerate structures for the JCF has been posted recently in [2] which strongly intersects and complement with our field of work [25].

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