Reserve Lifetime and Depletion in Exhaustible Resource Extraction Under Uncertainty

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FIRST VERSION: December 8, 2013

THIS VERSION: February 15, 2014

Abstract

This paper studies the implications for reserve lifetime and related quantities in a continuous time model of resource extraction under uncertainty. Both the resource price and the extracted amount are assumed to follow stochastic processes. Reserve lifetime is determined by the profit-maximizing firms optimal decision of when to close down production. We derive closed form expressions for the expected value and the probability distribution of reserve lifetime, the expected level of depletion, and the expected streams of discounted revenues and costs until closing.

JEL classification : Q30 G11 G13 M11 C61 C41 C44

Key words : Resource extraction, optimal stopping, reserve lifetime, depletion, real options, Brownian motion

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1 Introduction

The lifetime of a reserve is an important quantity to consider for resource extracting firms. Reserve lifetime is not only of interest to firm management but also to policy makers, analysts that monitor the firm and industry, and to the workers employed at the firm. This paper studies reserve lifetime, and related economic quantities, in a continuous time model of a resource extracting firm. In the model the lifetime of the reserve is a stochastic variable determined by the optimizing firm. One relevant measure of reserve lifetime in such a setting is naturally its expected value. Another measure is the probability that the reserve will remain open, or be closed, within a given time horizon. Other important economic measures related to reserve lifetime are the expected level of depletion at the abandonment time, and the total amounts of revenues and costs generated from operating the reserve. Since reserve lifetime is determined from economic considerations we use a real options model with a profit maximizing firm that optimizes its decision of when to terminate production and abandon the reserve. The model takes into account the uncertainties surrounding the firms profit flow and the option value of being able to close down production. It thus features many realistic and important features that influence the firms decision to abandon a particular reserve.

We derive closed form solutions for the optimal management rule of when to abandon, and the value of the reserve. Our main goal however is to extend the scope of the analysis of the model by deriving several new results regarding the implications of the optimal solution. First we derive, in closed form, the expected reserve lifetime given by the expected remaining time to abandonment. The distribution function of reserve lifetime is also derived and this gives us the probability that reserve lifetime will exceed a certain time span. We also derive a closed form expression for the expected total amount to be extracted until closing. All these quantities are of crucial interest to firm managers, analysts and policy-makers, and they allow for a richer analysis of the implications of the model. Finally, we derive expressions for other quantities of interest e.g. the total expected streams of revenues and costs during the reserve lifetime. As a particular application we study the total stream of revenue taxes paid by the firm until closing. This last measure is of interest both to the firm and to the tax setting authority.

We focus on the firms decision to abandon the reserve. In many cases the real option to abandon a particular reserve is the only relevant one. Many types of oil wells and off-shore oil rigs cannot be reopened (or put into mothball). The same is true for many types of mines where maintenance costs and the costs of reopening are too high to motivate mothballing.¹ For such

¹In a comprehensive study of Canadian copper mines Harchaoui and Lasserre (2001) e.g. find that the real options to mothball or reduce the extraction rate was never exercised by the firms included in their study. Moel and Tufano (2002), who study gold mines, find that indeed very few of the mines in their study were reopened

reserves the decision to close down and abandon is irreversible.

Most studies dealing with investment and management problems in the natural resource sector focus on commodity price risk as the sole source of uncertainty, see e.g. Brennan and Schwartz (1985), Paddock et al. (1988) and Bjerk Sund and Ekern (1990). Olsen and Stensland (1998) propose a model that incorporates the fact that the reserve is progressively eroded and that the extractable amount is uncertain. Abandonment options have been analysed in several earlier studies, see e.g. Myers and Majd (1990) and Dixit and Pindyck (1994) for general treatments and Olsen and Stensland (1998) and McCardle and Smith (1998) for application to resource extraction. The model we employ is very close to the one proposed in Olsen and Stensland (1998) but we make several relevant extensions. We show how to incorporate taxes and different discount factors for revenues and costs, and we also allow for a more flexible specification of the abandonment value. This paper differs from the preceding literature in that we show how to study the implications of the optimal management rule on important economic measures related to the length of the reserve lifetime. None of the above mentioned publications study these implications. With suitable reinterpretations of the economic variables and parameters our results should be useful in many applications not necessarily related to the extraction of a natural resource.

The rest of the paper is organized as follows. Section 2 introduces the model describing the firms problem. In Section 3 we provide the solution to the firms optimization problem. Section 4 contains our main results and presents a number of formulas for many different economically relevant measures related to reserve lifetime. Section 5 is devoted to a few numerical examples that highlight the impact of different parameters and assumptions on the economic measures we have derived. Section 6 concludes and all proofs are provided in the Appendix.

2 Resource extraction under uncertainty

We model a price taking firm that is in possession of a local reserve of an exhaustible resource e.g. a specific oil field or mine. In what follows we focus on the firms decision of when to optimally abandon the reserve and disregard any options to temporarily close production with the possibility of resuming operations in the future. Due to very high costs associated with reactivation and/or maintenance the decision to shut down is essentially irreversible for many, if not most, types of reserves. We therefore restrict our attention to the optimal close down decision which we assume is irreversible.

The firm will make optimal management decisions in an uncertain environment represented

after being closed.

by a complete filtered probability space $(\Omega, \mathbb{P}, \mathcal{F})$ that supports a standard 2-dimensional Brownian motion $W = (W^S, W^Q)'$. The price of the resource is given by a geometric Brownian motion (GBM) specified by the following stochastic differential equation (SDE)

$$dS_t = S_t (\mu_S dt + \sigma_S dW_t^S), \quad (1)$$

where $\mu_S \in \mathbb{R}$ and $\sigma_S > 0$. The assumption of a GBM governing the price of the resource is standard and it is employed in most of the related studies, see e.g. Brennan and Schwartz (1985), Olsen and Stensland (1998), McCardle and Smith (1998), Paddock et al (1989) and Bjerk Sund and Ekern (1990). More advanced models for commodity derivatives found in the asset pricing literature are also often built upon a GBM structure, see e.g. Schwartz and Smith (2000) and Casassus and Collin-Dufresne (2005). In this study we keep to the GBM specification (1) since it preserves the tractability of the analysis while still allowing for a stochastic evolution of the price of the resource.

It is assumed that the firm is extracting at its maximum capacity rate $q > 0$ and that the resource base evolves as

$$dQ_t = Q_t (-q dt + \sigma_Q dW_t^Q), \quad (2)$$

where $\sigma_Q > 0$. Reserve uncertainty is represented by the single standard Brownian motion W^Q . This makes reserve uncertainty independent of the uncertainty associated with the commodity price.² The amount qQ_t is extracted at each instant and subject to random disturbances driven by W^Q . The solution to (2) is given by

$$Q_t = Q_0 e^{-qt - \frac{1}{2}\sigma_Q^2 t + \sigma_Q W_t^Q}. \quad (3)$$

It follows from (3) that the resource will never be exhausted in finite time. The resource is only asymptotically exhaustible in the sense that $Q_t > 0$ for all $t < \infty$ and $\mathbb{P}(\{\lim_{t \rightarrow \infty} Q_t = 0\}) = 1$, see Øksendal (2000). This feature is not of central concern since under normal circumstances the time of abandonment is a finite number and hence production is terminated well before the resource is exhausted in the strict sense. It is at the time of abandonment that the reserve is exhausted in the economically relevant sense.

Extraction is modeled as an exponential decline which is a relatively standard assumption made in the literature, see e.g. Olsen and Stensland (1998) and Paddock et al. (1988). The

²This is a reasonable assumption in most applications since there is not any reason to expect the commodity price to be correlated with the reserve level in our context with a price taking firm operating a local reserve. However, if the reserve is a major contributor to the global supply of the resource, shocks to the production rate may indeed correlate with the price. It is straightforward to include a correlation between the price and reserve level. Closed form solutions would still be available but at the cost of having to deal with lengthier expressions.

stochastic part of the dynamics (2) can be motivated in several ways and has been proposed in Olsen and Stensland (1998). By modeling reserve uncertainty as proportional shocks to the reserve size captures the feature that reserve uncertainty should decline over time. It is plausible that uncertainty regarding the extractable amount of the resource is decreasing with time since more knowledge is accumulated over time and new technology may become available. Since Q decreases over time the proportional volatility specification in (2) is consistent with this feature. Another interpretation of the dynamics in (2) is that the random term represents unexpected deviations from a target, or mean, rate of extraction.

The firms profit flow at time t is given by

$$\Pi_t = \beta q S_t Q_t - c \quad (4)$$

where $c > 0$ denote flow costs which are assumed constant but may e.g. depend on q . The revenue from selling the extracted amount qQ_t at time t is given by $\beta q Q_t S_t$. We have introduced the parameter β to allow for a more flexible description of the profit flow. In the section on applications we will e.g. use β to represent different taxes. The parameter β can also be specified as $\beta = (q - \delta)/q$ with $\delta > 0$, such that while extraction reduces the reserve by the amount qQ the firm is only able to sell the amount $(q - \delta)Q$. The parameter δ would thus represent a wastage, or refining factor, that reduces the extracted amount that can be sold by an amount δQ that is lost in the extraction process.

The stochastic part of the instant revenue is given by $Z_t \equiv S_t Q_t$ which, again using Ito's formula, has the dynamics

$$dZ_t = Z_t \left(\mu_Z dt + \sigma_S dW_t^S + \sigma_Q dW_t^Q \right), \quad Z_0 = S_0 Q_0$$

where the drift of the revenue process, μ_Z , is given by

$$\mu_Z = \mu_S - q. \quad (5)$$

We will show that the firms optimal management rule, and the value of the reserve, can be expressed in terms of the single state variable Z . Before proceeding to the firms optimization problem we therefore collect some further useful results on this process. It will be convenient to express the dynamics of Z in terms of a single Brownian motion. For that purpose we note that we have the equality, in distribution,

$$\sigma_S W_t^S + \sigma_Q W_t^Q \stackrel{d}{=} \sigma_Z W_t^Z \quad (6)$$

where W^Z is a standard Brownian motion and

$$\sigma_Z = \sqrt{\sigma_S^2 + \sigma_Q^2}. \quad (7)$$

The asymptotic behavior of Z depends on the sign of $\mu_Z - \frac{1}{2}\sigma_Z^2$ and we will assume that the condition

$$\mu_Z - \frac{1}{2}\sigma_Z^2 < 0 \quad (8)$$

always holds. This condition is necessary for the firm's optimization problem to possess a well defined solution and it implies that $\mathbb{P}(\{\lim_{t \rightarrow \infty} Z_t = 0\}) = 1$, see Øksendal (2000), which is a reasonable economic assumption given that the reserve is exhaustible.³

3 Optimal reserve management

The objective of the firm is to maximize the value of the reserve given by the expected discounted future profit flows. Assuming the firm is risk neutral the reserve value at $t = 0$ is given by

$$V = \mathbb{E} \left[\int_0^\tau e^{-rt} \Pi_t dt \right] + \mathbb{E} [e^{-r\tau} (\eta_A Z_\tau - \theta_A)] \quad (9)$$

where $r > 0$ is the risk free interest rate and τ is the optimal time to abandon the reserve. When the firm abandons production it incurs the abandonment value $\eta_A S Q - \theta_A$ where η_A and θ_A are real numbers. This specification of the abandonment value is slightly more general than that found in Olsen and Stensland (1998).⁴

Remark 1 *The analysis also holds in a situation where revenues and costs are discounted at different rates; the revenue being discounted at a (risk-adjusted) capital cost, κ , while the deterministic cost flow is still discounted at the risk free interest rate. The value function in this situation reads:*

$$V = \mathbb{E} \left[\int_0^\tau (e^{-\kappa t} \beta q Z_t - e^{-rt} c) dt \right] + \mathbb{E} [e^{-\kappa \tau} \eta_A Z_\tau - e^{-r\tau} \theta_A].$$

This situation can be handled by rewriting the valuation into

$$V = \mathbb{E} \left[\int_0^\tau e^{-rt} (\beta q \tilde{Z}_t - c) dt \right] + \mathbb{E} [e^{-r\tau} (\eta_A \tilde{Z}_\tau - \theta_A C)].$$

³There are many parameter configurations that are consistent with condition (8). Sufficient conditions are e.g. that $\mu_S < \sigma_S^2/2$, or that $\mu_S < q$.

⁴When the firm values the profit stream in (9) by discounting at the risk free interest rate we are assuming that the firm is risk-neutral. This assumption is not necessary. We could also have assumed that S can be represented as a traded asset and that no risk premium is attached to reserve uncertainty. The valuation in (9) is then taken under an equivalent probability measure and consistent with any set of risk preferences. The drift terms μ_S should then be replaced by $\mu_S = r - \delta_S$ where δ_S is the spot convenience yield of S , see e.g. Schwartz (1997).

where we have defined $\tilde{Z}_t = \exp(-(\kappa - r)t)Z_t$. The analysis now carries through with the replacement $\mu_Z = \mu_S - q - (\kappa - r)$.⁵ We remark that using similar considerations, redefining the discount rate and the revenue process, it is also possible to allow for time dependent costs according to the specification $c(t) = c(0)\exp(g_c t)$. The parameter g_c is the growth rate in costs and may be negative or positive depending on the application. Note that in this case the extra condition $r > g_c$ must be satisfied.

Before proceeding we study reserve value when there is no option to abandon. For that purpose we define the following quantities

$$\lambda_Z = \frac{q\beta}{r - \mu_Z} \quad \text{and} \quad \lambda_C = \frac{c}{r}. \quad (10)$$

When there is no option to abandon we have that the expected discounted streams of revenues and costs can be expressed as

$$\mathbb{E} \left[\int_0^\infty e^{-rt} \beta q Z_t dt \right] = Z_0 \lambda_Z \quad \text{and} \quad \mathbb{E} \left[\int_0^\infty e^{-rt} c dt \right] = \lambda_C \quad (11)$$

which follows from straightforward calculation using the definitions in (10). Furthermore, in absence of the option to abandon the reserve value at $t = 0$ is given by

$$V_\infty \equiv \mathbb{E} \left[\int_0^\infty e^{-rt} \Pi_t dt \right] = \lambda_Z Z_0 - \lambda_C \quad (12)$$

which follows directly from (11). We shall throughout need to assume that the following inequalities hold

$$r - \mu_Z > 0 \quad ; \quad \lambda_C > \theta_A \quad ; \quad \lambda_Z > \eta_A \quad (13)$$

The first restriction in (13) is necessary to keep the present value of the expected stream of revenues positive and finite. The second restriction states that the cost of abandoning does not dominate the present value of running costs. Similarly, the third restriction states that the monetary income of abandoning does not dominate the present value of revenues.

We now turn to the firms optimization problem. In Appendix A we show that the optimal time to abandon the reserve is given by

$$\tau = \inf_{t \geq 0} \{Z_t \leq K_A\} \quad (14)$$

⁵It is straightforward to check that $\tilde{Z}$ becomes a GBM with drift $\mu_{\tilde{Z}} = \mu_S - q - (\kappa - r)$. The volatility σ_Z and the initial conditions are unchanged i.e. we have $\sigma_{\tilde{Z}} = \sigma_Z$ and $\tilde{Z}_0 = Z_0$.

where the threshold K_A is a real number. Note that we will assume that $Z_0 > K_A$ throughout otherwise the interpretation is that the firms immediately abandons. To streamline the presentation we define the following two quantities

$$\omega_0 = \frac{\ln(Z_0/K_A)}{\sigma_Z} > 0 \quad \text{and} \quad \omega_1 = \frac{\mu_Z - \frac{1}{2}\sigma_Z^2}{\sigma_Z} < 0 \quad (15)$$

which will appear frequently in the sequel. The barrier K_A is explicitly given by the expression

$$K_A = \frac{\gamma}{\gamma - 1} \frac{\lambda_C - \theta_A}{\lambda_Z - \eta_A} > 0, \quad (16)$$

with λ_Z and λ_C defined as in (10) and where $\gamma < 0$ is given by

$$\gamma = -\left(\frac{\omega_1}{\sigma_Z}\right) - \sqrt{\left(\frac{\omega_1}{\sigma_Z}\right)^2 + \frac{2r}{\sigma_Z^2}} < 0. \quad (17)$$

The value of the reserve at $t = 0$ is given by

$$V = \xi_A Z_0^\gamma + \lambda_Z Z_0 - \lambda_C, \quad (18)$$

where

$$\xi_A = \frac{\eta_A - \lambda_Z}{\gamma} K_A^{1-\gamma} > 0. \quad (19)$$

The optimal abandonment time τ is a random stopping time expressed as the first time the revenue process Z hits the level K_A . The barrier K_A depends in a highly non-linear fashion on the parameters of the firms optimization problem. However, the solution (16) is given on closed form and thus it is easily implemented and lends itself to straightforward numerical analysis. Note that the optimal reserve value (18) can be written as $V = \xi_A Z^\gamma + V_\infty$ showing that it consists of two parts; the long term value of the reserve, $V_\infty = (\lambda_Z Z - \lambda_C)$, plus the value of the option to abandon, $\xi_A Z^\gamma$.

The abandonment threshold $Z = K_A$ can be compared to the value $Z = \lambda_C/\lambda_Z$ that defines the point at which V_∞ turns negative. For $Z < \lambda_C/\lambda_Z$ the positive part of the reserve value V is only due to the value of the option to abandon. This option value is given by $\xi_A Z^\gamma$ and it is always positive and, since $\gamma < 0$, it tends to zero as $Z \rightarrow \infty$. The firm is thus prepared to accept some losses before making the irreversible and costly decision to close the reserve. Due to the uncertainties surrounding the firms profits there is always a chance that revenues will go up sufficiently to make profits positive again. The likelihood of this happening however decreases over time since the reserve is progressively emptied. These effects are balanced in the expression (16) that determine the location of K_A .

4 Applications

In this section we consider applications of the model which, despite being highly relevant, to our knowledge have not been previously studied in the literature on resource extraction under uncertainty before. The results we derive mostly rely on classical results in stochastic analysis. However, the proofs require some, more or less demanding, considerations before standard results may be applied. For that reason, and to keep the presentation more self-contained, we provide detailed proofs in Appendix A.

Related studies has focused the analysis of the optimal strategy on the location of the threshold at which the firm chooses to abandon. While the state variable Z has a simple and intuitive economic interpretation more precise information can be gained from the analysis of functionals of the optimal management rule. Such functionals include e.g. the expected reserve lifetime and the probability that the firm will abandon within a given time horizon. The optimal management rule derived in the previous section does not give much insight into how long the firm will keep the reserve open given the current values of revenues and costs. We argue that the expected time to abandon is an important additional tool for analyzing the firms operations. Expected reserve life is of course intuitively related to the location of the threshold K_A ; an increase in K_A will typically shorten the expected time to abandonment since the firm is expected to abandon earlier. However, this insight is not easily turned into an estimate of the actual amount of time left to abandonment. Neither does it allow for the probability of abandonment occurring within a certain time horizon. As we show below both the expected value and the probability distribution function for reserve lifetime are given by simple closed form expressions. We also show how other functionals of interest may be calculated. Examples include the expected level of depletion and the expected value of the streams of revenues and costs. Arguably, all these economic measures should be helpful in understanding the implications of the model and allow for a richer set of aspects of the optimal production rule to be evaluated.

4.1 Incorporating taxes

Before proceeding to the applications we demonstrate how to incorporate taxes in the model we have proposed. Suppose we wanted to analyse a model where the firms profit flow is given by

$$\Pi_t = (1 - \vartheta) [(1 - \rho)qZ_t - c]$$

With this formulation running profits are subject to the impact of two policy rates; an *income tax rate* denoted ϑ , and *revenue tax rate* denoted ρ . The solution to this problem has the same structure as the optimal solution to the original problem with a few minor changes. If we define

$$\beta = (1 - \vartheta)(1 - \rho) \quad ; \quad c = (1 - \vartheta)\tilde{c} \tag{20}$$

where $\tilde{c}$ are the actual flow costs (without the income tax), then the firms profit flow has the same structure as before. The optimal management rule is unchanged with the definitions of β and c made in (20). There are many applications of this extension. A policymaker may e.g. have a strategic objective of keeping production active for a time span that is different from that projected by the optimizing firm. Due e.g. to externalities (positive or negative) the socially optimal reserve life may be different from that targeted by a profit-maximizing firm. By choosing an appropriate tax rate, or subsidy, the government can influence the expected time operations remain active. Note that also the parameters η_A and θ_A that determine the abandonment value may also be redefined along the lines of (20) to incorporate different taxes. This is achieved by setting $\eta_A = (1 - \vartheta)(1 - \rho)\tilde{\eta}_A$ and $\theta_A = (1 - \vartheta)\tilde{\theta}_A$ where $\tilde{\eta}_A$ and $\tilde{\theta}_A$ are the parameters that determine abandonment values in the absence of taxes. Clearly, this redefinitions do not change the structure of the optimal management rule either.

4.2 Reserve lifetime

An important feature of the model we have employed is that very explicit results regarding the optimal abandonment time τ and expectations that depend on τ can be derived. As a first and useful result we have that the Laplace transform, or moment generating function, of τ defined as

$$L(u) = \mathbb{E} [e^{-u\tau}] , \quad u \geq 0, \quad (21)$$

has the explicit expression

$$L(u) = e^{\omega_0(\omega_1 - \sqrt{\omega_1^2 + 2u})} < 1. \quad (22)$$

This result is quite useful and we will use it a number of times. First of it allows us to derive the expected value of τ directly as

$$\mathbb{E} [\tau] = -L'(0) = \frac{\omega_0}{\omega_1}$$

Using the definitions of ω_0 and ω_1 we may conclude that

$$\mathbb{E} [\tau] = \frac{\ln(K_A/Z_0)}{\mu_Z - \frac{1}{2}\sigma_Z^2} > 0 \quad (23)$$

The expectation of τ gives direct and quantitative information regarding the expected time span during which the firm will actively explore the reserve.⁶ The proof of (22) is given in Appendix A.

⁶The formula in (22) allows for all moments of τ to be recovered. If we define $D_k(u)$ as the k 'th derivative of $L(u)$ we have the general relation $\mathbb{E} [\tau^k] = (-1)^k D_k(0)$.

A useful alternative, or complementary, measure of reserve lifetime is the probability $\mathbb{P}(\tau \leq t)$ that the reserve lifetime τ will be shorter than a given time horizon t . This probability is given by the distribution function for τ and as we show in Appendix A it has the closed form expression

$$\mathbb{P}(\tau \leq t) = N\left(\frac{-\omega_0 + \omega_1 t}{\sqrt{t}}\right) + e^{2\omega_0 \omega_1} N\left(\frac{-\omega_0 - \omega_1 t}{\sqrt{t}}\right), \quad (24)$$

where N denotes the cumulative distribution function of a standard $N(0, 1)$ random variable. Differentiating the above formula yields the following closed form expression for the density of the stopping time τ

$$f(t; \tau) = \frac{\omega_0}{\sqrt{t^3}} \varphi\left(\frac{\omega_0 - \omega_1 t}{\sqrt{t}}\right), \quad t \geq 0,$$

where $\varphi(x)$ is the probability density of a standard $N(0, 1)$ random variable. Given that conditions (8) and (13) holds this density integrates to one and we have $\mathbb{P}(\tau < \infty) = 1$ meaning that the reserve will be abandoned in finite time. The proof of equation (24) can be found in Appendix A.

4.3 Expected level of depletion

Another highly relevant quantity for firm management and policymakers to consider is how much of the resource that will ultimately be extracted or, equivalently, the final level of depletion defined as the amount left in the ground at the time of abandonment. The expected amount of the resource never to be extracted is given by $\mathbb{E}[Q_\tau]$. Conversely, the expected total amount actually to be extracted up until time τ is $Q_0 - \mathbb{E}[Q_\tau]$. One may also calculate expected depletion as the percentage of the initial reserve Q_0 that is left in the ground when the reserve is abandoned. This ratio is given by $\mathbb{E}[Q_\tau]/Q_0$. All these measures require the key quantity $\mathbb{E}[Q_\tau]$ to be calculated. Defining

$$\psi_1 = -\frac{1}{\sigma_Z} \left(\mu_Z + \sigma_Q^2 - \frac{1}{2}\sigma_Z^2 \right)$$

we show in Appendix A that we have the explicit expression

$$\mathbb{E}[Q_\tau] = Q_0 e^{\omega_0 (\psi_1 - \sqrt{\psi_1^2 + 2q})}. \quad (25)$$

In (25) we have $\omega_0 > 0$ and $\psi_1 - \sqrt{\psi_1^2 + 2q} < 0$ and hence that $\mathbb{E}[Q_\tau] < 1$ which is a natural requirement.

4.4 Expected stream of profits, revenues, costs and taxes

The final application that we consider is the calculation of the streams of expected discounted total profits, revenues and costs. As an application we show how to calculate the expected

discounted flow of taxes generated during the lifetime of the reserve. First we look at total profits. The expected value of the total stream of profits including the abandonment value is given by the value function V in (18). Given the optimal management rule to abandon at the stopping time τ the value function V can be written as

$$V = \beta q V_Z - c V_C + \eta_A R_Z - \theta_A R. \quad (26)$$

where we have defined

$$V_Z = \mathbb{E} \left[\int_0^\tau e^{-rs} Z_s ds \right] \quad ; \quad V_C = \mathbb{E} \left[\int_0^\tau e^{-rs} ds \right], \quad (27)$$

and

$$R_Z = \mathbb{E} [e^{-r\tau} Z_\tau] \quad ; \quad R = \mathbb{E} [e^{-r\tau}]. \quad (28)$$

The formula for V in (18) does not allow for the terms in (26) to be separately identified. However, having closed form expressions for these quantities should be of independent interest and can provide more insight into the properties of the optimal solution. Given the Laplace transform in (22) it is straightforward to calculate the key quantity R in (28). By the definition of (21) we have $R = L(r)$ and by the formula (22) we immediately find that

$$R = e^{\omega_0(\omega_1 - \sqrt{\omega_1^2 + 2r})} < 1. \quad (29)$$

The other quantities V_Z , V_C and R_Z can all be expressed in terms of R according to the following closed form expressions

$$V_Z = \frac{1}{r - \mu_Z} (Z_0 - K_A R) \quad ; \quad V_C = \frac{1}{r} (1 - R) \quad ; \quad R_Z = K_A R \quad (30)$$

with R given by (29). We refer to Appendix A for the proofs of these formulas. Using these expressions one can now e.g. calculate the expected total streams of revenues and costs as

$$\mathbb{E} \left[\int_0^\tau e^{-rs} q Z_s ds \right] = q V_Z \quad \text{and} \quad \mathbb{E} \left[\int_0^\tau e^{-rs} c ds \right] = c V_C$$

respectively where V_Z and V_C have the closed form expressions given in (30). Finally, the expected discounted abandonment value is given by

$$\mathbb{E} [e^{-r\tau} (\eta_A Z_\tau + \theta_A)] = \eta_A R_Z - \theta_A R = R (\eta_A K_A - \theta_A).$$

As an application of these results we consider how to calculate the expected total stream of discounted tax revenues. To simplify the presentation we assume that there is no income tax,

i.e. that $\vartheta = 0$, and that the firm only pays a revenue tax at the rate ρ . The total expected discounted stream of revenue taxes to be paid by the firm until closing is then given by

$$\Gamma = \rho \mathbb{E} \left[\int_0^\tau e^{-rs} q Z_s \right]. \quad (31)$$

From the results we have derived it is immediately clear that $\Gamma = \rho q V_Z$ where V_Z has the closed form expression in (30). There may also be a single tax payment at τ stemming from the abandonment value if it is taxed. The expected value of such a payment is given by $\rho R(\eta_K K_A - \theta_A)$.

5 Numerical examples

The model we have employed is very easy to use for analysis, comparative statics and scenario building. In this section we give some numerical examples of the economic measures we derived in the previous section. There are many interesting and relevant questions that can be addressed using the results we have derived. A main advantage of the model is that all important quantities are available on closed form. This fact greatly facilitates all kinds of analysis that one may wish to perform.

We start out with establishing a set of baseline parameters. We set $r = 0.04$, $q = 0.02$, $\mu_S = 0.05$, $\sigma_S = 0.3$ and $\sigma_Q = 0.05$. The reserve size is normalized to unity and we set $Q_0 = 1$. We also normalize initial spot price and set $S_0 = 1$. These parameters imply that $Z_0 = 1$ and the initial revenue is $qZ_0 = 0.02$. We further set $c = 0.04$ and assume that there is no income tax ($\vartheta = 0$) but a revenue tax of 30% ($\rho = 0.3$), which implies that $\beta = (1 - \rho) = 0.7$. Regarding the abandonment value we simply set $\eta_A = \theta_A = 0$ such that the abandonment value is zero. With these parameters the abandonment barrier is located at $K_A = 0.3109$, the firm is initially running a negative profit rate of $\Pi_0 = -0.0260$ and the expected remaining reserve lifetime is $\mathbb{E}[\tau] = 71.89$ years.

In figure 1 we have plotted the probability $\mathbb{P}(\tau \leq t)$ against t (in years) for three different levels of Z_0 . We set $Z_0 = 1$ (our baseline case), $Z_0 = 0.5$ and $Z_0 = 1.5$. The corresponding expected reserve lifetimes are 71.89, 29.2380 and 96.85449. Compared to the baseline case of $Z_0 = 1$ the effect on expected reserve lifetime of the lower value $Z_0 = 0.5$ is much stronger than that of the higher value Z_0 . This is also the case for the probabilities seen in figure 1 where the curve corresponding to $Z_0 = 0.5$ increases much more rapidly compared to the curves corresponding to the higher values of $Z_0 = 1$ and $Z_0 = 1.5$. We conclude that the state variable Z does not only affect the level of the probability curves $\mathbb{P}(\tau \leq t)$, but it can also have a strong impact on the shape of the curve. As a numerical example we have that the probability that the reserve will be closed before $t = 20$ years is close to 75% when $Z_0 = 0.5$, while it does not exceed 45% for the other two cases of $Z_0 = 1$ and $Z_0 = 1.5$.

INSERT FIGURE 1 AROUND HERE

We end this section with three examples of how the location of Z_0 and the level of the revenue tax ρ influences a selection of the quantities we have derived. First we illustrate how expected reserve lifetime and expected depletion depends on the state variable Z at different levels of the revenue tax rate ρ . In figure 2 we have plotted reserve lifetime as a function of the state variable Z for three different levels of ρ . We set $\rho = 0$, $\rho = 0.3$ and $\rho = 0.6$. Note that each curve in figure 2 is plotted against different ranges of Z_0 . This is because as we change ρ the abandonment barrier K_A , and consequently $\mathbb{E}[\tau]$, changes as well. We have only plotted $\mathbb{E}[\tau]$ for values of Z_0 that are above the barrier K_A such that the reserve is in operation. The barriers for our choice of revenue taxes are: $K_A = 0.2176$ when $\rho = 0$, $K_A = 0.3109$ when $\rho = 0.3$, and $K_A = 0.5441$ when $\rho = 0.6$.

INSERT FIGURE 2 AROUND HERE

Expected reserve lifetime is an increasing function of Z_0 . In our base case, with $Z_0 = 1$ and $\rho = 0.3$, expected reserve lifetime is $\mathbb{E}[\tau] = 71.89$ years. At $Z_0 = 1$ an increase in the revenue tax from $\rho = 0.3$ to $\rho = 0.6$ will decrease $\mathbb{E}[\tau]$ with around 34 years while setting $\rho = 0$ increases $\mathbb{E}[\tau]$ with around 22 years. As can be seen in figure 2 these effects are relatively uniform across the different levels of Z_0 .

Our second example concerns the depletion ratio defined as

$$D = \mathbb{E} \left[\frac{Q_\tau}{Q_0} \right]$$

This ratio measures the percentage of the initial reserve that will be left in the ground when the site is abandoned. In figure 3 we perform the same analysis as we did for the expected reserve lifetime and plot the ratio D as a function of Z for different values of the revenue tax rate ρ . As in figure 2 each curve is plotted against the range of Z_0 -values for which $Z_0 > K_A$.

INSERT FIGURE 3 AROUND HERE

Figure 3 shows that similar to $\mathbb{E}[\tau]$ the effects on D of altering the tax rate ρ is close to uniform across the levels of Z_0 . Decreasing the tax rate ρ from 30% to 0% increases expected reserve life by around 22 years at $Z_0 = 1$. The corresponding decrease for the depletion ratio is from 0.5412 to 0.4487, i.e. instead of an expected 54% of the initial reserve left in the ground, this number falls to around 45%. Increasing the tax rate ρ from 0.3 to 0.6 instead increases the expected amount left in the ground to around 72.6%.

Finally, we make a similar illustration for the expected discounted stream of tax revenues Γ defined in (31). Using equations (30) and (29) we can write Γ as

$$\Gamma = \rho q V_Z = \rho q \left(Z_0 - K_A e^{\sigma_Z^{-1} \ln(Z_0/K_A) (\omega_1 - \sqrt{\omega_1^2 + 2r})} \right). \quad (32)$$

where we used the definition of ω_0 in (15) to emphasize the dependency on Z_0 . Recall that we have set the abandonment value to zero and hence there is no contribution to revenue taxes from this value in our example. In figure 4 we have plotted Γ as a function of Z_0 for the two different values of $\rho = 0.3$ and $\rho = 0.6$.

INSERT FIGURE 4 AROUND HERE

Figure 4 highlights the fact that setting a high tax rate not necessarily leads to an increase in Γ . As can be seen in the graph; for low levels of Z_0 the value of the tax revenue stream is higher at the lower tax rate $\rho = 0.3$. The reason is that a higher tax rate has two opposing effects; it raises the tax revenue obtained per unit time but at the same time it shortens the expected reserve lifetime since it reduces the profitability of the reserve. At low levels of Z_0 the second effect, that reduces Γ , is stronger since the reserve is already operating at low levels of revenues. At sufficiently high levels of Z_0 the opposite is true. Note that at the baseline parameters we have chosen the quantity Γ depends almost linearly on Z_0 . This need not always be the case and equation (32) shows that the dependency in general is non-linear.

6 Concluding remarks

In this paper the properties of reserve lifetime, and related quantities, are studied in a model of a profit maximizing firm who optimally determines when to close down production. The model is formulated in continuous time where both the price of the resource and the reserve processes are given by stochastic processes. We solve the profit maximizing firms problem of when to optimally abandon extraction from a given reserve.

The main goal of the paper is to study economic quantities of importance that are functions of the optimal abandonment time. We derive closed formulas for the expected value of reserve lifetime, the probability density of reserve lifetime, the expected level of depletion at the abandonment time, and the expected value of the discounted stream of tax revenues generated by the firms operations. Despite being highly relevant these quantities have, to the best of our knowledge, not been studied in the context of resource extraction under uncertainty before. Our results have relevance for many other situations where it is of interest to study the expected lifetime, and related quantities, of an economic project under uncertainty.

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APPENDIX A: Proofs

Proof of the optimal management rule and reserve value. The proof of the optimal abandonment time and the value of the reserve is relatively standard. Proofs in closely related models can be found elsewhere in the literature, see e.g. Olsen and Stensland (1998), who prove analogue results for the management rule and reserve value in a model that is similar to ours. To keep the presentation more self-contained, and since our model slightly differs from the mentioned references, we provide a sketch of the proof of the optimal management rule and the reserve value.

In general the firms maximization problem is to find the optimal value function V and abandonment time τ such that at time t we have

$$V(t, S, Q) = \sup_{\tau \in \mathcal{T}} J^\tau(t, S, Q) \quad (\text{A.1})$$

where

$$J^\tau(t, S, Q) = \mathbb{E} \left[\int_0^\tau e^{-r(u+t)} \Pi_u du - e^{-r(t+\tau)} (\eta_A Z_\tau - \theta_A) \right]$$

and where $\mathcal{T}$ denotes the set of admissible stopping times τ .

From Theorem 10.4.1 in Øksendal (2000) the solution is characterized by a region D for the state space (t, S, Q) where it is optimal to continue extraction. A basic requirement of the value function V is that it satisfies the partial differential equation

$$\mathcal{L}V(t, S, Q) + e^{-rt} \beta q (SQ - c) = 0 \quad (\text{A.2})$$

for $(t, S, Q) \in D$. Here we have introduced the derivative operator $\mathcal{L}$, associated with the state variables (t, S, Q) and operating on a function $\phi(t, S, Q)$. The operator $\mathcal{L}$ is defined by

$$\mathcal{L}\phi = \frac{\partial \phi}{\partial t} + \mu_S S \frac{\partial \phi}{\partial S} - qQ \frac{\partial \phi}{\partial Q} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 \phi}{\partial S^2} + \frac{1}{2} \sigma_Q^2 Q^2 \frac{\partial^2 \phi}{\partial Q^2}.$$

Before proceeding with the specification of the continuation region D and the appropriate boundary conditions we show how to express equation (A.2) in terms of the single state variable $Z \equiv SQ$. We guess that the solution to equation (A.2) can be expressed as

$$V(t, Z) = e^{-rt} \phi(Z) \quad (\text{A.3})$$

for some function $\phi(Z)$. Taking the required derivatives of this proposed solution and substituting into the original equation (A.2) it is straightforward to show that it reduces to

$$-r\phi(Z) + Z\phi'(Z)\mu_Z + \phi''(Z)Z^2 \frac{1}{2} \sigma_Z^2 + \beta qZ - c = 0. \quad (\text{A.4})$$

where μ_Z and σ_Z are defined in (5) and (7). This is an ordinary differential equation for the function ϕ in the single state variable Z . A general solution to (A.4) is given by

$$\phi(Z) = \xi_A Z^{\gamma^-} + \varrho Z^{\gamma^+} + \lambda_Z Z - \lambda_C \quad (\text{A.5})$$

where λ_Z and λ_C are defined in (10), and ξ_A and ϱ are constants to be determined. The exponents $\gamma^\pm$ are the solutions

$$\gamma^\pm = \frac{1}{2} - \frac{\mu_Z}{\sigma_Z^2} \pm \sqrt{\left(\frac{\mu_Z}{\sigma_Z^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma_Z^2}}. \quad (\text{A.6})$$

to the quadratic equation $h(\gamma) = 0$ where

$$h(\gamma) = \frac{1}{2}\sigma_Z^2\gamma(\gamma - 1) + \gamma\mu_Z - r. \quad (\text{A.7})$$

The roots (A.6) satisfy $\gamma^- < 0$ and $\gamma^+ > 1$. This follows from noting that $h(\gamma) \rightarrow \infty$ as $|\gamma| \rightarrow \infty$ revealing that $h(\gamma)$ has a minimum. Further we have $h(0) = -r < 0$ and $h(1) = \mu_Z - r < 0$ by assumption (13). Combining these arguments is clear that $h(\gamma)$ has a negative minimum and that there is a positive and a negative root. Since $h(1) < 0$ it also follows that $\gamma^+ > 1$.

The two last terms of $\phi(Z)$ in (A.5) expresses the additional value that derives from the option to abandon. Since the value of this option must tend to zero as Z tends to infinity we require that

$$\lim_{Z \rightarrow \infty} (\varrho Z^{\gamma^+} + \xi_A Z^{\gamma^-}) = 0 \quad (\text{A.8})$$

for the solution to make economic sense. Since $\gamma^+ > 1$ condition (A.8) cannot be fulfilled unless $\varrho = 0$ and we thus arrive at our final solution:

$$\phi(Z) = \xi_A Z^{\gamma^-} + \lambda_Z Z - \lambda_C. \quad (\text{A.9})$$

The first term in (A.9) captures the option value of abandoning where the constant ξ_A must be determined from additional boundary conditions.

The reduction in state variables carries over to the continuation region D which can be expressed as $D^Z = \{Z : Z > K_A\}$ for some $K_A > 0$. The optimal time to abandon will then be

$$\tau = \inf \{t \geq 0 : Z_t \leq K_A\}, \quad (\text{A.10})$$

i.e. it is the first time the process Z exits from D^Z . To find the option value multiple ξ_A and the abandonment threshold K_A we make use of the condition that the value function V should be continuous and continuously differentiable as a function of Z at the point of abandonment.

This requirement is met by the so called value matching and smooth pasting conditions that we can state as

$$\phi(K_A) = (\eta_A K_A - \theta_A C) \quad ; \quad \phi'(K_A) = \eta_A \quad (\text{A.11})$$

These conditions define a system of equations that we can now solve for ξ_A and the barrier K_A explicitly using the solution (A.9). This gives the expressions stated in the text. Finally, defining $\gamma \equiv \gamma^-$ and $V \equiv V(0, Z_0)$ we have that the reserve value at $t = 0$ is given by

$$V = \xi_A Z_0^\gamma + \lambda_Z Z_0 - \lambda_C$$

as stated in the text. It can be verified that the solution we have constructed satisfies the conditions of Theorem 10.4.1 in Øksendal (2000) from which it follows that the proposed value function (A.3) and stopping time (A.10) are optimal. ■

Proof of the formula for the Laplace transform of τ . Recall that the stopping time τ is defined as $\tau = \inf \{t \geq 0 | Z_t \leq K_A\}$. Since we are only interested in situation where $Z_0 > K_A$ we have $\tau = \inf \{t \geq 0 | Z_t = K_A\}$. Using the equality in distribution (6) together with the definitions (5) and (7) we first note that $Z_t \stackrel{d}{=} Z_0 \exp \left\{ \left(\mu_Z - \frac{1}{2} \sigma_Z^2 \right) t + \sigma_Z W_t^Z \right\}$ where W_t^Z is a standard Brownian motion. Next, we define the process $X_t = \ln(Z_t/Z_0)$. We then have

$$X_t = \left(\mu_Z - \frac{1}{2} \sigma_Z^2 \right) t + \sigma_Z W_t^Z, \quad X_0 = 0,$$

which is a Brownian motion with drift. Furthermore, the stopping time τ can be expressed as the first time the process X hits the level $X_\tau = \ln(K_A/Z_0)$. Next, we define the process

$$H_a(t) = e^{aX_t - p(a)t} \quad (\text{A.12})$$

where

$$p(a) = \left(\mu_Z - \frac{1}{2} \sigma_Z^2 \right) a + \frac{1}{2} \sigma_Z^2 a^2. \quad (\text{A.13})$$

It is straightforward to verify that the process H_a is a martingale with $\mathbb{E}[H_a(t)] = H_a(0)$. By the Optional Stopping Theorem, see e.g. Chang (2004), p. 263, we also have the stronger relation

$$\mathbb{E}[H_a(\tau)] = H_a(0).$$

It follows from the definition of X_t that we have $H_a(0) = 1$. We further find that

$$\mathbb{E}[H_a(\tau)] = e^{aX_\tau} \mathbb{E} \left[e^{-p(a)\tau} \right] = e^{a \ln(K_A/Z_0)} \mathbb{E} \left[e^{-p(a)\tau} \right].$$

where we used that $X_\tau = \ln(K_A/Z_0)$ is non-stochastic. The equality (A.12) now yields the expression

$$\mathbb{E} \left[e^{-p(a)\tau} \right] = e^{a \ln(Z_0/K_A)} \quad (\text{A.14})$$

which follows after some rearrangement of the terms. Next we set $p(a) = u$. From (A.13) the roots to this quadratic equation are

$$a^\pm = \frac{1}{\sigma_Z} \left(\omega_1 \pm \sqrt{\omega_1^2 + 2u} \right).$$

where we used the definition of ω_1 in (15). There is one negative and one positive root; $a^- < 0$ and $a^+ > 0$. Since $\mathbb{E}[\exp(-u\tau)] < 1$ for $u \geq 0$, we must choose the negative root $a = a^-$. Substituting $p(a) = u$ and $a = a^-$ in (A.14) we finally arrive at the expression

$$L(u) = e^{\omega_0(\omega_1 - \sqrt{\omega_1^2 + 2u})},$$

where we used the fact that $a \ln(Z_0/K_A) = \omega_0(\omega_1 - \sqrt{\omega_1^2 + 2u})$ using the definitions of ω_0 and ω_1 in (15). ■

Proof of the formula for the distribution function of τ . To derive the distribution function $\mathbb{P}(\tau \leq r)$ we note that

$$\mathbb{P}(\tau > t) = \mathbb{P}\left(\min_{s \in [0, t]} (Z_s) > K_A\right)$$

Again using the equality in distribution (6) together with (5) and (7) we first note that $Z_t \stackrel{d}{=} Z_0 \exp\left\{(\mu_Z - \frac{1}{2}\sigma_Z^2)t + \sigma_Z W_t^Z\right\}$ where W_t^Z is a standard Brownian motion. We can now write

$$\mathbb{P}\left(\min_{s \in [0, t]} (Z_s) > K_A\right) = \mathbb{P}\left(\min_{s \in [0, t]} \left\{ \left(\mu_Z - \frac{1}{2}\sigma_Z^2\right)t + \sigma_Z W_t^Z \right\} > \ln(K_A/Z_0)\right).$$

With the definitions of ω_0 and ω_1 we can write this probability as

$$\mathbb{P}\left(\min_{s \in [0, t]} (Z_s) > K_A\right) = \mathbb{P}\left(\min_{s \in [0, t]} \{-\omega_1 t + W_t^Z\} > -\omega_0\right)$$

The last probability is a standard result in stochastic analysis and from Corollary B.3.4. in Musiela and Rutkowski (1998) we have

$$\mathbb{P}\left(\min_{s \in [0, t]} \{-\omega_1 t + W_t^Z\} > -\omega_0\right) = N\left(\frac{\omega_0 - \omega_1 t}{\sqrt{t}}\right) - e^{2\omega_0\omega_1} N\left(\frac{-\omega_0 - \omega_1 t}{\sqrt{t}}\right) \quad (\text{A.15})$$

where $N(x)$ denotes the cumulative distribution function of a standard $N(0, 1)$ random variable. The probability $\mathbb{P}(\tau \leq t)$ can now be obtained as

$$\begin{aligned} \mathbb{P}(\tau \leq t) &= 1 - \mathbb{P}(\tau > t) \\ &= 1 - \mathbb{P}\left(\min_{s \in [0, t]} \{-\omega_1 t + W_t^Z\} > -\omega_0\right) \\ &= N\left(\frac{-\omega_0 + \omega_1 t}{\sqrt{t}}\right) + e^{2\omega_0\omega_1} N\left(\frac{-\omega_0 - \omega_1 t}{\sqrt{t}}\right) \end{aligned}$$

where we used (A.15) and the property $N(-x) = 1 - N(x)$ in the last step. This concludes the proof. ■

Proof of the formula for the expected extracted amount. First we have from (3) that

$$\mathbb{E}[Q_\tau] = Q_0 \mathbb{E}[e^{-q\tau} M_\tau] \quad (\text{A.16})$$

where we have defined $M_t = \exp\left\{-\frac{1}{2}\sigma_Q^2 t + \sigma_Q W_t^Q\right\}$. Note that the process M is a $\mathbb{P}$ -martingale and thus that $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for $s < t$. In particular it holds that $\mathbb{E}[M_t] = M_0 = 1$. Next, we use the Girsanov theorem, see Karatzas and Shreve (1996), to construct a new measure $\bar{\mathbb{P}}$ that is equivalent to $\mathbb{P}$ on $\mathcal{F}_\tau$. This will allow us to calculate the expected value in (A.16) without having to deal with the joint density of M_τ and $\exp(-q\tau)$. To achieve this we define the new measure $\bar{\mathbb{P}}$ on $\mathcal{F}_\tau$ by $d\bar{\mathbb{P}}/d\mathbb{P} = M_t$. The filtration $\mathcal{F}_\tau$ is the σ -algebra of measurable sets A such that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$. From the Girsanov theorem it now follows that the stochastic process $\bar{W} = (\bar{W}^S, \bar{W}^Q)'$ defined by the relation

$$dW_t = (0, \sigma_Q)' dt + d\bar{W}_t \quad (\text{A.17})$$

is a standard Brownian motion under $\bar{\mathbb{P}}$. Note that the change of measure only affects the Brownian component W^Q such that $dW_t^Q = \sigma_Q dt + d\bar{W}_t^Q$. The other component W^S is unaffected with $dW_t^S = d\bar{W}_t^S$. After having obtained these results we may now write

$$\mathbb{E}[Q_\tau] = Q_0 \mathbb{E}[e^{-q\tau} M_\tau] = Q_0 \bar{\mathbb{E}}[e^{-q\tau}] \quad (\text{A.18})$$

where $\bar{\mathbb{E}}$ denotes the expectation operator under $\bar{\mathbb{P}}$. What is left is now to calculate the final expectation in (A.18). This amounts to calculating the Laplace transform of the stopping time τ under $\bar{\mathbb{P}}$. For that purpose we need the dynamics of Z under $\bar{\mathbb{P}}$ and using (A.17) we find that

$$\frac{dZ_t}{Z_t} = (\mu_Z + \sigma_Q^2) dt + \sigma_S d\bar{W}_t^S + \sigma_Q d\bar{W}_t^Q.$$

where $\mu_Z = \mu_S - q$ as before. We may again use the relation $\sigma_Z \bar{W}_t^Z \stackrel{d}{=} \sigma_S \bar{W}_t^S + \sigma_Q \bar{W}_t^Q$ with σ_Z still defined by (7). What is left is to calculate the Laplace transform $\bar{\mathbb{E}}[\exp(-q\tau)]$. The rest of the proof therefore proceeds in the same way as the above proof for the formula in (22). The only differences are that the drift of $\ln(Z)$ is changed to $\mu_Z + \sigma_Q^2 - (1/2)\sigma_Z^2$, that ω_1 is replaced by ψ_1 defined in (4.3), and that we set $u = q$. We can hence conclude that

$$\mathbb{E}[Q_\tau] = Q_0 e^{\omega_0(\psi_1 - \sqrt{\psi_1^2 + 2q})},$$

which is the formula presented in (25). ■

Proof of the formulas in equations (29) and (30). We have already calculated the formula for $R = \mathbb{E}[\exp(-r\tau)]$ from (22) by setting $u = r$. Next, we calculate the formula for V_Z defined in (27). First, note that

$$V_Z = \mathbb{E} \left[\int_0^\tau e^{-rt} Z_t dt \right] = \mathbb{E} \left[\int_0^\infty e^{-rt} Z_t dt \right] - \mathbb{E} \left[\int_\tau^\infty q e^{-rt} Z_t dt \right] \quad (\text{A.19})$$

where we have

$$\mathbb{E} \left[\int_0^\infty e^{-rt} Z_t dt \right] = \frac{1}{r - \mu_Z} Z_0.$$

The second integral in (A.19) can be written

$$\mathbb{E} \left[\int_\tau^\infty e^{-rt} Z_t dt \right] = \mathbb{E} \left[e^{-r\tau} \int_0^\infty e^{-rs} Z_{\tau+s} ds \right] \quad (\text{A.20})$$

using a change of variables. Next, we define the process $X_s = Z_{\tau+s}$. The initial condition of this process is $X_0 = Z_\tau = K_A$ and it is independent of $\mathcal{F}_\tau$, see e.g. Chang (2004) p.265, or Steele (2001), p.108. Conditioning on the filtration $\mathcal{F}_\tau$ we now have from (A.20) that

$$\begin{aligned} \mathbb{E} \left[\int_\tau^\infty e^{-rt} Z_t dt \right] &= \mathbb{E} \left[e^{-r\tau} \mathbb{E} \left[\int_0^\infty e^{-rs} X_s ds \middle| \mathcal{F}_\tau \right] \right] \\ &= \mathbb{E} \left[e^{-r\tau} \mathbb{E} \left[\frac{1}{r - \mu_Z} Z_\tau \middle| \mathcal{F}_\tau \right] \right] \\ &= \frac{1}{r - \mu_Z} K_A \mathbb{E} [e^{-r\tau}] \end{aligned}$$

Using the definition of R this establishes the formula for V_Z in (30). The proof of the formula for V_C in (30) rests on identical considerations. However for this quantity we may perform the more direct calculation

$$\begin{aligned} V_C &= \mathbb{E} \left[\int_0^\tau e^{-rt} dt \right] \\ &= \mathbb{E} \left[\frac{1}{r} (1 - e^{-r\tau}) \right] \\ &= \frac{1}{r} (1 - \mathbb{E} [e^{-r\tau}]) \\ &= \frac{1}{r} (1 - R), \end{aligned}$$

which establishes the expression for V_C in (30). Finally, the expression for R_Z found in (30) can be calculated as

$$R_Z = \mathbb{E} [e^{-r\tau} Z_\tau] = \mathbb{E} [e^{-r\tau} \mathbb{E} [Z_\tau | \mathcal{F}_\tau]] = \mathbb{E} [e^{-r\tau}] K_A = K_A R$$

which is the formula presented for R_Z in (30) and concludes the proof. ■

Figures

Figure 1: The probability, $\mathbb{P}(\tau \leq t)$, plotted as a function of t for three different values of Z_0 .

Figure 2: Expected reserve lifetime, $\mathbb{E}[\tau]$, plotted as a function of Z_0 for three different values of the revenue tax rate ρ .

Figure 3: Expected depletion ratio, $D = E[Q_\tau/Q_0]$, as a function of Z_0 for three different values of the revenue tax rate ρ .

Figure 4: Expected stream of total discounted revenue taxes, Γ , as a function of Z_0 for two different levels of the revenue tax rate ρ .