From the Axiom of Choice
to Tychonoff’s Theorem

Gustav Hörngren

Hösterminen 2014

Handledare: Holger Schellwat
Examinator: Yang Liu

Självständigt arbete, 15 hp
Matematik, C – nivå, 76 – 90 hp
Abstract

A topological space $X$, is shown to be compact if and only if every net in $X$ has a cluster point. If $s$ is a net in a product $\prod_{\alpha \in A} X_{\alpha}$, where each $X_{\alpha}$ is a compact topological space, then, for every subset $B$ of $A$, such that the restriction of $s$ to $B$ has a cluster point in the partial product $\prod_{\alpha \in B} X_{\alpha}$, it is found that the restriction of $s$ to $B \cup \{\gamma\}$ – extending $B$ by one element $\gamma \in A \setminus B$ – has a cluster point in its respective partial product $\prod_{\alpha \in B \cup \{\gamma\}} X_{\alpha}$, as well.

By invoking Zorn’s lemma, the whole of $s$ can be shown to have a cluster point. It follows that the product of any family of compact topological spaces is compact with respect to the product topology. This is Tychonoff’s theorem.

The aim of this text is to set forth a self contained presentation of this proof. Extra attention is given to highlight the deep dependency on the axiom of choice.
# Contents

1 Introduction 5
   1.1 Background ................................. 5
   1.2 Chapter outline .............................. 5

2 Sets 7
   2.1 Basic notation and terminology ............. 7
   2.2 Relations and Mappings ..................... 9
   2.3 Generalized unions, intersections and cartesian products .... 11
   2.4 The Axiom of choice ........................ 14
   2.5 Orderings and Zorn’s lemma ................. 16

3 Point Set Topology 19
   3.1 Basics ....................................... 19
   3.2 Compactness .................................. 23
   3.3 The product topology ......................... 24

4 Nets 27
   4.1 Directed sets ................................ 27
   4.2 Nets and Subnets ............................. 28
   4.3 Nets and Compactness ......................... 32
      4.3.1 A proof of Tychonoff’s theorem .......... 34

A A proof of Zorn’s lemma 41
   A.1 The Bourbaki fixed-point theorem ............ 41
      A.1.1 A proof of the Bourbaki fixed-point theorem .... 41
   A.2 From the Bourbaki fixed-point theorem to Zorns lemma .... 45
Chapter 1

Introduction

1.1 Background

Tychonoff’s theorem states that the generalized cartesian product of any family of compact topological spaces is compact with respect to the induced product topology. The proof was first stated and proved in 1930 by Andrey N. Tychonoff.

In 1992 American mathematician Paul Chernoff, published an article [1] in which he introduced a new way to prove the theorem. According to his survey, there are now – counting his own contribution – four different proofs of the theorem.

Chernoff’s proof uses the theory of nets (a generalization of the theory of sequences) and how compactness of topological spaces finds a description in terms of the theory of nets. Although this way of describing compactness had been known for many decades, its application to a proof of Tychonoff’s theorem was yet unknown.

The aim of this essay is to present a detailed version of Chernoff’s proof and the mathematics that it depends upon, thus making the article [1] available to readers with no or little prior knowledge of set theory and point set topology. In fact, with the exception of a couple of illustrating examples, the prerequisites for reading this essay – besides the capability of reading mathematical discourse – are none.

1.2 Chapter outline

Chapter 2 Here we develop the basic notions of the set theoretical language required for the developments in later chapters. In particular many results there are dependent on the axiom of choice, which is why we have treated it extensively. Finally Zorn’s lemma is stated; a proof is supplied in Appendix A.
Chapter 3 The point of this chapter is to define the product topology, which is a topology on the generalized cartesian product of a family of topological spaces, and to find a base for it.

Chapter 4 Here we develop the theory of nets to the point where we can prove that the definition of compactness for a topological space is equivalent to every net in that space having a cluster point. The chapter is concluded by giving a reworked, and more detailed version of the proof in [1], of Tychonoff’s theorem.

The set theory draws mainly from [4], except the proof of Zorn’s lemma, which is based on [5, pages 878–884]. The topology in Chapter 3 is mainly from [2, chapter 4] and the material about nets draws from [1], [2, chapter 4] and [3, chapter 2]. Finally, besides [1], the main proof is also found in [2, pages 136–137].
Chapter 2

Sets

The aim of this chapter is to present the bits of set-theoretical language relevant to later chapters.

2.1 Basic notation and terminology

A set is a collection of objects (or things). If $x$ is an object that belong to the set $A$, then $x$ is said to be a member of $A$ (‘$x$ is contained in $A$’ or ‘$A$ contains $x$,’ are synonymous). We write this as

$$x \in A.$$ 

If $x$ is an object that is not a member of $A$, we may write this as

$$x \notin A.$$ 

If $A$ and $B$ are sets such that whenever $x \in B$, we have $x \in A$, then $B$ is said to be a subset of $A$. This we write as

$$B \subset A \text{ (or } A \supset B).$$

Notice that every set is a subset of itself.

The empty set, denoted by $\emptyset$, is defined as the set with no members. The empty set is subset of every set. Indeed, for a set $A$ to not have $\emptyset$ as a subset the statement:

whenever $x \in \emptyset$, we have $x \in A$ \hspace{1cm} (2.1)

must be false. But this is impossible. In fact, the negation of it reads:

there exists $x \in \emptyset$ such that $x \notin A$. \hspace{1cm} (2.2)
But, by definition, it does not exist any \( x \in \emptyset \), so (2.2) is clearly false. Hence (2.1) is true.

Furthermore, we will write

\[ A = B \]

for \( A \subset B \) and \( B \subset A \),

with the meaning of ‘for’ taken in the sense of ‘as an abbreviation of’ or ‘instead of.’ In other words, two sets \( A \) and \( B \) are defined as equal if they are subsets of each other. Thus, in general, if we want to prove that two sets \( A \) and \( B \), are equal, we need to prove that \( A \) is a subset of \( B \) and \( B \) is a subset of \( A \).

As a special case, we have sets whose members can be listed as finite or infinite arrays of characters (each character representing a member of the set). For instance, we have the set of the objects \( x_1, \ldots, x_n \) or the set of all positive integers \( 1, 2, 3, \ldots \). The former we will write as

\[ \{ x_1, \ldots, x_n \}, \] \hspace{1cm} (2.3)

and the latter as

\[ \{ 1, 2, 3, \ldots \}. \] \hspace{1cm} (2.4)

An object \( y \), is a member of the set (2.3) if and only if \( y \) is equal to at least one of the listed \( x_j \). Thus, for the listing in (2.3) and (2.4); order is irrelevant. Also note that duplicates may be yanked without changing the set, that is, if two different characters represent the same object, we can safely remove one of them and the shorter list of characters will still represent the set it did before the removal; to illustrate we have

\[ \{ x_1, \ldots, x_n, a \} = \{ x_1, \ldots, x_n \} \]

whenever \( a = x_k \) for some \( k = 1, \ldots, n \) (the same holds in the infinite case).

A set is said to be finite if it can be written as in (2.3), otherwise it is said to be infinite.

We may introduce sets by a phrase of the form:

\[ \text{the set of all } x, \text{ such that } \ldots \ldots \] \hspace{1cm} (2.5)

where for any given object \( x \) (as soon as we “plug it in” to ‘\( \ldots \ldots \)’) the expression ‘\( \ldots \ldots \)’ represents a statement about \( x \), that if true, qualifies \( x \) as a member and otherwise as a non-member of the set (2.5).

As a convenient abbreviation for (2.5) we have the piece of notation

\[ \{ x : \ldots \ldots \} \]

where in particular the ‘\( : \)’-sign may be read as ‘such as.’
Clearly \( y \in \{ x : \ldots x \ldots \} \) if and only if the statement ‘\( \ldots y \ldots \)’ is true.

As a tweak of the \( \{ x : \ldots x \ldots \} \)-notation, it is common to write part of the qualifying statement before the ‘:’-sign in. For example, we may write the set of all odd, positive integers as

\[
\{ x \in \mathbb{Z}_+ : x \text{ is odd} \},
\]

where of course \( \mathbb{Z}_+ \) represents the set \( \{ 1, 2, 3, \ldots \} \).

If \( A \) is any set, then the power set of \( A \), denoted by \( \mathcal{P}(A) \), is defined as the set of all subsets of \( A \). Or in short:

\[
\mathcal{P}(A) = \{ B : B \subseteq A \}.
\]

We round up this section by defining three binary operations on the class of sets. Given a pair of sets \( A \) and \( B \), we define the union of \( A \) and \( B \), denoted \( A \cup B \), as

\[
A \cup B = \{ x : x \in A \text{ or } x \in B \},
\]

the intersection of \( A \) and \( B \), denoted \( A \cap B \), as

\[
A \cap B = \{ x : x \in A \text{ and } x \in B \},
\]

and the difference between \( A \) and \( B \), denoted \( A \setminus B \), as

\[
A \setminus B = \{ x : x \in A \text{ and } x \notin B \}.
\]

If \( A \cap B = \emptyset \), we say that \( A \) and \( B \) are mutually disjoint.

### 2.2 Relations and Mappings

Let \( A \) and \( B \) be sets. If \( a \in A \) and \( b \in B \) then we can form a new kind of object \( (a, b) \), called the ordered pair of \( a \) and \( b \). Two ordered pairs \( (a, b) \) and \( (a', b') \) are defined as equal if and only if \( a = a' \) and \( b = b' \).

The cartesian product \( A \times B \) of \( A \) and \( B \), is defined as

\[
A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}.
\]

Let \( A \) and \( B \) be sets. A relation from \( A \) to \( B \) is a subset of \( A \times B \).

Let \( R \) be a relation from \( A \) to \( B \). It is common to write

\[
'xRy' \text{ for } '(x, y) \in R',
\]

as in ‘\( 3 \leq 4 \)’ where \( \leq \) is regarded as the usual less than or equal to relation, from \( \mathbb{Z}_+ \) to itself.
The set \( \{ x \in A : xRy \text{ for at least one } y \in B \} \) is called the domain of \( R \) and the set \( \{ y \in B : xRy \text{ for at least one } x \in A \} \) is called the range of \( R \).\(^1\)

A relation \( R \subset A \times B \), is said to be a mapping from \( A \) into \( B \) if:

(i) the domain of \( R \) includes the whole of \( A \),

(ii) whenever \( xRy \) and \( xRy' \), then \( y = y' \). 

In other words, \( R \) is a mapping if for every \( x \in A \), there exists a unique \( y \in B \) such that \( xRy \).

If \( R \) is a mapping and \( xRy \), then we say that \( y \) is the value of the argument \( x \), or that \( y \) is the value of \( R \) at \( x \).

Terminology. It is common to refer to mappings as functions or transformations, but we will almost exclusively use the word ‘mapping.’

Notation. There are some common pieces of notation used to distinguish those characters that represent mappings. In particular, we will denote the set of all mappings from \( A \) into \( B \) by \( B^A \). Hence, we can write \( f \in B^A \) to state that \( f \) is a mapping from \( A \) to \( B \). The same may also be conveyed by writing

\[ f : A \to B. \]

If \( f \in B^A \), it is custom, and often convenient, to denote its value at \( \alpha \) by \( f(\alpha) \). If, however, the associated value of an argument \( \alpha \) is denoted by, say \( \beta_\alpha \), we may draw attention to this by writing

\[ f : A \to B \text{ such that } \alpha \mapsto \beta_\alpha. \]

In this case, we don’t even have to tie a mapping to any particular name, such as \( f \), but we are perfectly alright referring to it by phrases like:

the mapping \( \alpha \mapsto \beta_\alpha \) in \( B^A \)

or,

the mapping \( A \to B : \alpha \mapsto \beta_\alpha \)

thus refraining completely from representing it by a special character, and instead concentrate on the association that makes up the mapping. In fact, this will be the default throughout this text.

The compound formulæ’ \( \alpha \mapsto \beta_\alpha \) may be read as ‘\( \alpha \) maps to \( \beta_\alpha \).’

\(^1\)Also common in the literature on the subject is to use the name codomain here.
Let \( f \in B^A \) and let \( E \subset A \). Then we define the image – denoted by \( f(E) \) – of \( E \) under \( f \) as

\[
f(E) = \{ f(x) : x \in E \}.
\]

The set \( f(A) \), is simply called the image of \( f \).

Now let \( f \in B^A \), as above, but this time let \( E \subset B \). Then we define the inverse image – denoted by \( f^{-1}(E) \) – of \( E \) under \( f \) as

\[
f^{-1}(E) = \{ x \in A : f(x) \in E \}.
\]

If \( \alpha \mapsto \beta_\alpha \) is a mapping in \( B^A \) and \( \beta \mapsto \gamma_\beta \) is a mapping in \( C^B \), then the association

\[
\alpha \mapsto \beta_\alpha \mapsto \gamma_\beta
\]

is a mapping in \( C^A \). A chain of mappings like this is called a composite mapping. If \( f \in B^A \) and \( g \in C^B \), their composition \( \alpha \mapsto f(\alpha) \mapsto g(f(\alpha)) \) is denoted by

\[
g \circ f.
\]

That is, \( g \circ f(\alpha) = g(f(\alpha)) \) for all \( \alpha \in A \).

Let \( f : A \to X \) and \( B \subset A \). Then the mapping \( B \to X \) such that \( x \mapsto f(x) \) is denoted by \( f|B \) and is called the restriction of \( f \) to \( B \).

The following definition provides a key tool to our proof of Tychonoff’s theorem.

**2.1 Definition.** Let \( A \) be a subset of \( B \). And let \( f \in X^A \) be a mapping from \( A \) into some set \( X \). Then a mapping \( g \in X^B \) is said to be an extension of \( f \) if \( g|A = f \).

**2.3 Generalized unions, intersections and cartesian products**

Often it is helpful to distinguish sets whose members are also sets, by referring to them as families or collections (of sets).

**2.2 Definition.** Let \( \mathcal{F} \) be a family of sets. Then we define

\[
\bigcup \mathcal{F} = \{ x : x \in F \text{ for at least one } F \in \mathcal{F} \},
\]

\[
\bigcap \mathcal{F} = \{ x : x \in F \text{ for all } F \in \mathcal{F} \},
\]

called the union and the intersection of \( \mathcal{F} \) respectively.
Note that the union \( A \cup B \), in the binary sense, of two sets \( A \) and \( B \), is the union of the family \( \{A, B\} \) (similarly for binary intersections). Hence it is justified to speak of the unions and intersections in Definition 2.2 as \textit{generalized} versions of the binary operations with the same name.

Besides “regular” families, we will find it useful to talk about so called \textit{indexed families}.

\begin{definition}
Let \( \mathcal{F} \) be a family and let \( A \) be any set. A mapping
\[ A \rightarrow \mathcal{F} \text{ such that } \alpha \mapsto X_\alpha \]  
(2.6)
is called an \textit{indexed family}. We will denote the indexed family (2.6) by \( \{X_\alpha\}_{\alpha \in A} \).
\end{definition}

Notice the difference between writing \( \{X_\alpha\}_{\alpha \in A} \) and \( \{X_\alpha : \alpha \in A\} \). The latter is actually the image of the former.

Often we will refer to indexed families as, simply, families, but the subscript notation \( \{X_\alpha\}_{\alpha \in A} \) will clarify that we talk about an indexed family (whenever this makes any difference.)

\begin{definition}
We define the \textit{union} \( \bigcup_{\alpha \in A} X_\alpha \), of an indexed family \( \{X_\alpha\}_{\alpha \in A} \), as the union of its image. That is
\[ \bigcup_{\alpha \in A} X_\alpha = \bigcup \{X_\alpha : \alpha \in A\}. \]
Similarly we define the \textit{intersection} \( \bigcap_{\alpha \in A} X_\alpha \), of \( \{X_\alpha\}_{\alpha \in A} \) as
\[ \bigcap_{\alpha \in A} X_\alpha = \bigcap \{X_\alpha : \alpha \in A\}. \]
\end{definition}

\textit{Notation.} It is often convenient to denote the union and intersection of a non-indexed family \( \mathcal{F} \) as \( \bigcup_{F \in \mathcal{F}} F \) and \( \bigcap_{F \in \mathcal{F}} F \), respectively. When doing so, we are actually viewing \( \mathcal{F} \) as the indexed family
\[ \mathcal{F} \rightarrow \mathcal{F} \text{ such that } F \mapsto F, \]
that is \( \{F\}_{F \in \mathcal{F}} \).

Strictly speaking, \( \mathcal{F} \) and \( \{F\}_{F \in \mathcal{F}} \) are not equal (in fact \( \{F\}_{F \in \mathcal{F}} \) is a subset of \( \mathcal{F} \times \mathcal{F} \)), but their respective unions and intersections are equal, which accounts for most practical purposes.

\begin{proposition}[De Morgan’s laws]
Let \( X \) be a set and let \( \mathcal{F} \subset \mathcal{P}(X) \). Then we have:
\end{proposition}
Proof. The proof is straight-forward.

\[ \bigcup_{F \in \mathcal{F}} (X \setminus F) = \{ x : x \in X \setminus F \text{ for at least one } F \in \mathcal{F} \} = \{ x \in X : x \notin \bigcap_{F \in \mathcal{F}} F \} = X \setminus (\bigcap_{F \in \mathcal{F}} F), \]

which is (a). Similarly for (b) we have

\[ \bigcap_{F \in \mathcal{F}} (X \setminus F) = \{ x : x \in X \setminus F \text{ for all } F \in \mathcal{F} \} = \{ x \in X : x \notin \bigcup_{F \in \mathcal{F}} F \} = X \setminus (\bigcup_{F \in \mathcal{F}} F), \]

which concludes the proof. □

2.6 Definition. We define the **generalized cartesian product** \( \prod_{\alpha \in A} X_\alpha \), of an indexed family \( \{X_\alpha\}_{\alpha \in A} \), as

\[ \prod_{\alpha \in A} X_\alpha = \left\{ f \in \left( \bigcup_{\alpha \in A} X_\alpha \right)^A : f(\alpha) \in X_\alpha \text{ for all } \alpha \in A \right\}. \]

Notice that if \( X_\alpha = X \) for all \( \alpha \) in Definition 2.6, the we have \( \prod_{\alpha \in A} X_\alpha = X^A \).

The generalized cartesian product is central to Tychonoff’s theorem, which says something about the generalized cartesian products of families of topological spaces, satisfying certain conditions.

A few things are worth noting: For instance, how do the generalized cartesian product of the family \( \{X_1, X_2\} \) relate to the (non-generalized) cartesian product \( X_1 \times X_2 \), of its members? We have

\[ \prod_{k \in [1,2]} X_k = \{ (\{1, x_1\}, (2, x_2)) : (x_1, x_2) \in X_1 \times X_2 \}. \]
Indeed, for \( f \) to be a mapping in \( (X_1 \cup X_2)^{\{1,2\}} \) such that \( f(1) \in X_1 \) and \( f(2) \in X_2 \), \( f \) must have the form \( \{(1,x_1),(2,x_2)\} \) where \( x_1 \in X_1 \) and \( x_2 \in X_2 \). So although \( \prod_{k \in \{1,2\}} X_k \) and \( X_1 \times X_2 \) aren’t equal, they do stand in a one-to-one relationship to each other.

A mapping \( f \in \prod_{\alpha \in A} X_\alpha \) is sometimes called a choice function on the family \( \{X_\alpha\}_{\alpha \in A} \) (because its image \( f(A) \), is what we might get if we were to choose one element from each of the \( X_\alpha \). Every choice function thus corresponds to one way of making such a choice) and for any \( \alpha \in A \), it’s value \( f(\alpha) \), at \( \alpha \), is called the \( \alpha \)-coordinate of \( f \).

Next, note that if \( \{X_\alpha\}_{\alpha \in A} \) is any indexed family, then for every \( \beta \in A \), there exists a unique mapping
\[
\prod_{\alpha \in A} X_\alpha \to X_\beta : f \mapsto f(\beta),
\]
that projects every \( f \in \prod_{\alpha \in A} X_\alpha \) on its \( \beta \)-coordinate. These mappings are important and go under a special name.

**2.7 Definition.** Let \( \{X_\alpha\}_{\alpha \in A} \) be an indexed family. For all \( \beta \in A \) define
\[
\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta \text{ such that } f \mapsto f(\beta).
\]
The family \( \{\pi_\alpha\}_{\alpha \in A} \), of mappings are referred to as the coordinate mappings on \( \{X_\alpha\}_{\alpha \in A} \).

Thus we may write the \( \alpha \)-coordinate of some \( f \in \prod_{\alpha \in A} X_\alpha \) as \( \pi_\alpha(f) \).

If \( U_\alpha \subset X_\alpha \), then the inverse image \( \pi_\alpha^{-1}(U_\alpha) \), of the respective coordinate mapping, is the set of choice functions on \( \{X_\alpha\}_{\alpha \in A} \) whose \( \alpha \)-coordinate is in \( U_\alpha \).

To every family \( \{X_\alpha\}_{\alpha \in A} \) of non-empty sets, there exists at least one choice-function. This is, in fact, a version of the axiom of choice, to which the following section is devoted.

### 2.4 The Axiom of choice

**2.8 Definition.** Let \( \mathcal{A} \) be a non-empty collection of non-empty, mutually disjoint sets. A set \( B \subset \bigcup \mathcal{A} \) is said to be a selection on \( \mathcal{A} \) if \( B \) contains exactly one member from each \( A \in \mathcal{A} \) (that is, for all \( A \in \mathcal{A} \), \( B \cap A \) is a one-member set).

Hence, if we have a family of non-empty, mutually disjoint sets, a selection on this family is obtained by simply choosing exactly one member of each set. It appears, thus, that whenever we have a non-empty family \( \mathcal{A} \), of non-empty, mutually
disjoint sets, we can always get a selection $S$ on $\mathcal{A}$. Indeed, we could just go ahead and choose one member from each set in $\mathcal{A}$ (after all, the sets are non-empty) and the set of members thus chosen is a selection on $\mathcal{A}$. Most literature in algebra and analysis – including [2, 3, 5] – accepts this as a valid move, the formal statement of which is the so called \textit{axiom of choice}.

\textbf{The Axiom of Choice.} If $\mathcal{A}$ is a non-empty collection of non-empty, mutually disjoint sets, then there exists a selection on $\mathcal{A}$.

In certain pathological cases the axiom of choice can cause the existence of some rather strange mathematical objects and phenomena. A discussion of these cases is beyond the scope of this text. One relatively simple example can be found in [2, pages 19–21].

Problematic results, like the one in [2], only arise when $\mathcal{A}$ is infinite, and seems to have something to do with the fact that there isn’t a good answer to the question: \textit{How}, exactly, are we to choose one member from each set in an infinite family? Psychologically speaking, the bringing about of a selection from an infinite family of mutually disjoint sets, without constructing it in a “reasonably explicit fashion,” [2, page 17] feels a bit artificial, as opposed to when $\mathcal{A}$ is finite; then, at least, we can somehow imagine the choosing in question from a finite array of characters representing $\mathcal{A}$’s members.

There are some nuances of opinion as to what extent and under what conditions it is permissible to make the “choice move” on $\mathcal{A}$ and receive a selection to pass on. See [6, pages 490–492] for a brief introduction to this discussion.

We will use the axiom of choice several times in the sequel. In fact, several of the results that we will prove are not possible without it. But we will always be explicit about using the axiom of choice when doing so. Moreover we will use it in a different form.

2.9 Proposition (The multiplicative axiom). If $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ is a non-empty family of non-empty sets, then $\prod_{\alpha \in \mathcal{A}} X_\alpha \neq \emptyset$.

Proof. Put

$$\mathcal{A} = \{ \{a\} \times X_\alpha : \alpha \in \mathcal{A} \}.$$ 

Note that for any pair of $\alpha, \beta \in \mathcal{A}$, if $\alpha = \beta$ then obviously $\{a\} \times X_\alpha = \{\beta\} \times X_\beta$; else, if $\alpha \neq \beta$, then $\{a\} \times X_\alpha$ and $\{\beta\} \times X_\beta$ cannot have any element in common. Indeed, for all members $(\xi, x) \in \{a\} \times X_\alpha$, there is only one choice for $\xi$ (that is, $\xi = \alpha$) so we can not have $(\xi, x) \in \{\beta\} \times X_\beta$, since this would render $\xi = \beta$, and thereby $\alpha = \beta$. Thus it is clear that $\mathcal{A}$ is a non-empty collection of non-empty, mutually disjoint sets.

Hence, by the axiom of choice there exists a selection, $S \subset \bigcup \mathcal{A}$, on $\mathcal{A}$. 

15
It turns out that $S \in \prod_{\alpha \in A} X_\alpha$. In fact, note that $S \subset A \times (\bigcup_{\alpha \in A} X_\alpha)$. Thus it only remains to verify that for each $\alpha \in A$ there exists $x_\alpha \in X_\alpha$ such that $(\alpha, x_\alpha) \in S$, and if $x \in \bigcup_{\alpha \in A} X_\alpha$ with $(\alpha, x) \in S$ then $x = x_\alpha$.

Accordingly, let $\alpha \in A$. Since $S$ is a selection on $A$, $S \cap (\{\alpha\} \times X_\alpha)$ is a one-member set. Hence there exists $x_\alpha \in X_\alpha$ such that $(\alpha, x_\alpha) \in S$. Now, suppose $x \in \bigcup_{\alpha \in A} X_\alpha$ such that $(\alpha, x) \in S$. By $(\alpha, x) \in S \subset \bigcup A$, we get some $\beta \in A$ such that $(\alpha, x) \in (\{\beta\} \times X_\beta)$. Clearly $\beta = \alpha$, so $(\alpha, x) \in S \cap (\{\alpha\} \times X_\alpha)$, which is a one-member set that already contains $(\alpha, x_\alpha)$. Hence $(\alpha, x_\alpha) = (\alpha, x)$, rendering $x = x_\alpha$, which concludes the proof. □

Bertrand Russell used Proposition 2.9 instead of the axiom of choice and called it the multiplicative axiom [7, page 536]. In practice Russell’s version of the axiom of choice is certainly more useful (although less primitive). Clearly it implies the axiom of choice. Hence the multiplicative axiom and the axiom of choice mutually equivalent.

2.5 Orderings and Zorn’s lemma

2.10 Definition. Let $S$ be a set. A relation $\leq$ on $S$ is called a partial ordering on $S$ if

(i) $x \leq x$, for all $x \in S$,

(ii) If $x \leq y$ and $y \leq x$, then $x = y$, for all $x, y \in S$,

(iii) If $x \leq y$ and $y \leq z$, then $x \leq z$, for all $x, y, z \in S$.

We also say that $S$ is partially ordered (by $\leq$).

Terminology. The three listed properties (i), (ii) and (iii) that make up a partial ordering in Definition 2.10, are commonly called, the reflexive, anti-symmetric and transitive property, respectively.

We will occasionally adopt the notation: ‘$x < y$’ for ‘$x \leq y$ and $x \neq y$’.

2.11 Definition. A partial ordering $\leq$ on a set $S$, is called a total ordering if for all $x, y \in S$, we have either $x \leq y$ or $y \leq x$. Then $S$ is said to be totally ordered.

Notice that if $T$ is a subset of some partially ordered set $S$, then $T$ becomes partially ordered as well. This by letting the partial ordering on $S$ induce an ordering on $T$. On top of being partial, the induced ordering may very well be total. Below there will be plenty occasion to speak of totally ordered subsets of some partially ordered set.
2.12 Definition. Let $T$ be a subset of some partially ordered set $S$. An element $b \in S$ is called an upper bound for $T$ in $S$ if for all $x \in T$, we have $x \leq b$. If also $b \leq b'$ for all upper bounds $b'$, then $b$ is called a least upper bound (for $T$ in $S$).

2.13 Definition. Let $S$ be partially ordered by $\leq$ and let $m \in S$. We will say that $m$ is a maximal element of $S$ if whenever $x \in S$ such that $m \leq x$, then $m = x$.

Notice that neither upper bounds nor maximal elements necessarily are unique (there may be several), but that least upper bounds are.

2.14 Definition. A partially ordered set $S$ is said to be inductively ordered if every totally ordered subset $T$ of $S$, has an upper bound $b$ in $S$. Moreover, we say that $S$ is strictly inductively ordered if $b$ is a least upper bound for $T$ in $S$.

Zorn’s lemma. If $S$ is a non-empty, inductively ordered set, then $S$ has a maximal element.

Zorn’s lemma is a critical constituent of every known proof of Tykhonoff’s theorem as well as many other results in algebra and analysis (notably the Hahn-Banach theorem [2, page 158] or that every vector space has a basis.)

In Appendix A, we have set forth a proof of the statement that Zorn’s lemma follows from the axiom of choice.
Chapter 3

Point Set Topology

3.1 Basics

3.1 Definition. Let $X$ be a non-empty set. A family $\mathcal{O} \subset \mathcal{P}(X)$ is called a topology on $X$ if

(i) $\emptyset, X \in \mathcal{O}$,

(ii) $\bigcup \mathcal{F} \in \mathcal{O}$ for all $\mathcal{F} \subset \mathcal{O}$,

(iii) $\bigcap \mathcal{F} \in \mathcal{O}$ for all finite $\mathcal{F} \subset \mathcal{O}$.

The ordered pair $(X, \mathcal{O})$ is called a topological space.

Terminology. If $(X, \mathcal{O})$ is a topological space, then the sets $E \in \mathcal{O}$ are said to be open (in $X$) and sets of the form $X \setminus E$, where $E \in \mathcal{O}$, are said to be closed (in $X$.) If the topology $\mathcal{O}$ is understood or unspecified, we may drop reference to it and speak of $X$, simply, as a topological space.

Let $X$ be a topological space. Notice that the sets $\emptyset$ and $X$ are both open and closed in $X$.

3.2 Example. For every non-empty set $X$, $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are both topologies on $X$ called the trivial and discrete topology respectively.

3.3 Proposition. Let $X$ be a topological space and let $E \subset X$.

(a) If $E$ is closed in $X$, then $X \setminus E$ is open in $X$.

(b) If $X \setminus E$ is open in $X$, then $E$ is closed in $X$. 
Proof. If \( E \) is closed in \( X \), then there exists an open set \( F \) in \( X \), such that \( E = X \setminus F \), so

\[
X \setminus E = X \setminus (X \setminus F)
\]

\[
= \{ x : x \in X \text{ and } x \notin X \setminus F \}
\]

\[
= \{ x \in X : \neg (x \notin F) \}
\]

\[
= F,
\]

which is open in \( X \). Hence (a).

To prove (b), assume that \( X \setminus E \) is open in \( X \). Then \( X \setminus (X \setminus E) \) is closed in \( X \), but with identical reasoning as in the proof of (a), we get

\[
X \setminus (X \setminus E) = E.
\]

Hence (b). \( \Box \)

3.4 Proposition. Let \( X \) be a topological space. If \( \mathcal{F} \) is a family of closed sets then \( \bigcap \mathcal{F} \) is also closed in \( X \). Furthermore, if \( \mathcal{F} \) also is finite, then \( \bigcup \mathcal{F} \) is closed in \( X \) as well.

Proof. We begin by proving the first part of the statement. Accordingly, let \( \mathcal{F} \subset \mathcal{P}(X) \) be a family of sets such that each \( F \in \mathcal{F} \) is closed in \( X \).

We want to show that \( \bigcap_{F \in \mathcal{F}} F \) is closed in \( X \). By Proposition 3.3.b, this will follow if we can show that \( X \setminus (\bigcap_{F \in \mathcal{F}} F) \) is open in \( X \). But by Proposition 3.3.a, \( X \setminus F \) is open in \( X \) for all \( F \in \mathcal{F} \). Hence \( \bigcup_{F \in \mathcal{F}} (X \setminus F) \) is open in \( X \). Now by Proposition 2.5.a we get

\[
\bigcup_{F \in \mathcal{F}} (X \setminus F) = X \setminus (\bigcap_{F \in \mathcal{F}} F),
\]

thus concluding (a).

To prove the second part, assume that \( \mathcal{F} \), in addition, is finite.

We want to show that \( \bigcup_{F \in \mathcal{F}} F \) is closed in \( X \). But by Proposition 3.3.b, this will follow if we can show that \( X \setminus (\bigcup_{F \in \mathcal{F}} F) \) is open in \( X \). But, since \( X \setminus F \) is open in \( X \) for all \( F \in \mathcal{F} \) and \( \mathcal{F} \) is finite, we get \( \bigcap_{F \in \mathcal{F}} (X \setminus F) \) to be open in \( X \) and thus \( X \setminus (\bigcup_{F \in \mathcal{F}} F) \) is open in \( X \), since

\[
\bigcup_{F \in \mathcal{F}} (X \setminus F) = X \setminus (\bigcup_{F \in \mathcal{F}} F)
\]

by Proposition 2.5.b. \( \Box \)

3.5 Definition. Let \( X \) be a topological space and \( A \subset X \). Then we define
(a) $A^o = \bigcup\{ E : E \subset A \text{ and } E \text{ is open in } X \}$,

(b) $\overline{A} = \bigcap\{ E : E \supset A \text{ and } E \text{ is closed in } X \}$.

$A^o$ called the interior of $A$ and $\overline{A}$ is called the closure of $A$.

Let $X$ be a topological space and let $A \subset X$. Note that $A^o$ is open in $X$ and that $\overline{A}$ is closed in $X$. In fact, $A^o$ is the largest open set that is a subset of $A$ and $\overline{A}$ is the smallest closed set that has $A$ as a subset.

3.6 Definition. Let $X$ be a topological space and let $x \in X$. A set $U \subset X$ is called a neighborhood of $x$ if $x \in U^o$.

3.7 Definition. Let $X$ be a topological space and let $A \subset X$. A point $x \in X$ is called an accumulation point of $A$ if for every neighborhood $U$ of $x$, we have $A \cap (U^o \setminus \{x\}) \neq \emptyset$. The set of accumulation points of $A$ will be denoted by $acc(A)$.

3.8 Proposition. Let $X$ be a topological space and let $A \subset X$. Then $\overline{A} = A \cup acc(A)$.

Proof. First we will show that $\overline{A} \subset A \cup acc(A)$. Notice that this will follow if we can show that $X \setminus [A \cup acc(A)] \subset X \setminus \overline{A}$. Consequently, let $x \in X \setminus [A \cup acc(A)]$. We want to show that $x \notin \overline{A}$. But since $x \notin acc(A)$ there must exist an open set $U$, such that $x \in U$ and $A \cap (U \setminus \{x\}) = \emptyset$. Hence, since $x \notin A$, we get $A \cap U = \emptyset$. In particular, $X \setminus U$ is both closed and $A \subset X \setminus U$, which renders $\overline{A} \subset X \setminus U$. Now, since $x \in U$, we get $x \notin X \setminus U$ and thus $x \notin \overline{A}$ as well.

It remains to show that $A \cup acc(A) \subset \overline{A}$. Again we will do this by showing $X \setminus \overline{A} \subset X \setminus [A \cup acc(A)]$.

Accordingly, let $x \in X \setminus \overline{A}$. We need to show that $x \notin A$ and $x \notin acc(A)$. But $x \notin A$ follows immediately, since we have $\overline{A} \supset A$ and $x \notin \overline{A}$. To show that $x \notin acc(A)$ it is sufficient to find a neighborhood $U$ of $x$, such that $A \cup U = \emptyset$. But note that $X \setminus \overline{A}$ is open and contains $x$. Hence it is a neighborhood of $x$ that in addition does not intersect $A$, which is exactly what we were looking for.

3.9 Proposition. Let $\mathcal{O}$ be a family of topologies on some set $X$. Then $\bigcap \mathcal{O}$ is a topology on $X$.

Proof. Since $\emptyset, X \in \mathcal{O}$ for all $\mathcal{O} \in \mathcal{O}$, we have $\emptyset, X \in \bigcap \mathcal{O}$ and if $\mathcal{F} \subset \bigcap \mathcal{O}$ then $\mathcal{F} \subset \mathcal{O}$ for all $\mathcal{O} \in \mathcal{O}$. Hence $\bigcup \mathcal{F} \in \mathcal{O}$ for all $\mathcal{O} \in \mathcal{O}$, which is to say that $\bigcup \mathcal{F} \in \bigcap \mathcal{O}$. In the same way we get $\bigcap \mathcal{F} \in \bigcap \mathcal{O}$ for all finite $\mathcal{F} \subset \bigcap \mathcal{O}$. Thus $\mathcal{O}$ is a topology on $X$.

3.10 Definition. Let $X$ be any non-empty set and let $\mathcal{E} \subset \mathcal{P}(X)$. Then the topology

$$\bigcap\{ \mathcal{O} : \mathcal{O} \supset \mathcal{E} \text{ and } \mathcal{O} \text{ is a topology on } X \}$$

is called the topology on $X$ generated by $\mathcal{E}$.
Let $X$ be a non-empty set and let $\mathcal{E} \subset \mathcal{P}(X)$. Note that – if nothing else – $\mathcal{E}$ is always a subset of the discrete topology. Hence, the intersection in Definition 3.10 is always non-empty, so the topology on $X$ generated by $\mathcal{E}$ always exists.

3.11 Definition. Let $\mathcal{O}$ be a topology on a set $X$. A family $\mathcal{B} \subset \mathcal{O}$ is said to be a base for $\mathcal{O}$ if for all $U \in \mathcal{O} \setminus \{\emptyset\}$ there exists $\mathcal{F} \subset \mathcal{B}$, with $U = \bigcup \mathcal{F}$.

3.12 Theorem. Let $X$ be a non-empty set and let $\mathcal{E} \subset \mathcal{P}(X)$. Then the set

$$\mathcal{B} = \{ \bigcap \mathcal{F} : \mathcal{F} \subset \mathcal{E} \text{ and } \mathcal{F} \text{ is finite} \} \cup \{X\}$$

is a base for the topology on $X$ generated by $\mathcal{E}$.

Proof. Denote the topology on $X$ generated by $\mathcal{E}$ by $\mathcal{O}(\mathcal{E})$ and put

$$\mathcal{O} = \{ U \subset X : \text{for all } x \in U \text{ there exists } V \in \mathcal{B} \text{ with } x \in V \subset U \}.$$ 

If we can prove the following:

(a) $\mathcal{O}$ is a topology on $X$,

(b) $\mathcal{B}$ is a base for $\mathcal{O}$,

(c) $\mathcal{E} \subset \mathcal{O} \subset \mathcal{O}(\mathcal{E})$,

in that order, we’re done. Indeed, if (c) then $\mathcal{O} = \mathcal{O}(\mathcal{E})$ (provided (a)).

It is immediately clear that $\emptyset, X \in \mathcal{O}$. Also note that $\bigcup \mathcal{F} \in \mathcal{O}$ for all $\mathcal{F} \subset \mathcal{O}$. Indeed, if $x \in \bigcup \mathcal{F}$, then $x \in F$ for some $F \in \mathcal{F}$, but this $F$ is a member of $\mathcal{O}$, so $x \in V \subset F \subset \bigcup \mathcal{F}$ for some $V \in \mathcal{B}$. Thus, to complete the proof of (a), we need to show that $\mathcal{O}$ is closed under finite intersections.

Accordingly let $U_1, \ldots, U_n \in \mathcal{O}$ and let $x \in U_1 \cap \cdots \cap U_n$. We need to show that there exists $V \in \mathcal{B}$ with $x \in V \subset U_1 \cap \cdots \cap U_n$. But for each $j = 1, \ldots, n$ we have some $V_j \in \mathcal{B}$ with $x \in V_j \subset U_j$. Now, since each $V_j$ is a finite intersection of sets in $\mathcal{E}$, the set $V = V_1 \cap \cdots \cap V_n$ must be a finite intersection of sets in $\mathcal{E}$ as well, so $V \in \mathcal{B}$. Moreover, we have $x \in V \subset U_1 \cap \cdots \cap U_n$, and (a) follows.

To prove (b), first note that $\mathcal{B} \subset \mathcal{O}$. Indeed, if $V \in \mathcal{B}$, then for all $x \in V$, we have $x \in V \subset V$. Now, let $U \in \mathcal{O} \setminus \{\emptyset\}$. We need to show that there exists $\mathcal{F} \subset \mathcal{O}$ with $U = \bigcup \mathcal{F}$. But note that for every $x \in U$ the sets

$$A_x = \{ V \in \mathcal{B} : x \in V \subset U \}$$

are non-empty. Hence by Proposition 2.9 there exists

$$\Gamma \in \prod_{x \in U} A_x$$

22
so \( \{ \Gamma(x) : x \in U \} \subset \mathcal{B} \), and \( U = \bigcup_{x \in U} \Gamma(x) \). Which is (b).

In order to prove (c), first note that if \( E \in \mathcal{E} \) then, by \( \{ E \} \) being a finite subset of \( \mathcal{E} \), we get \( \bigcap \{ E \} = E \in \mathcal{B} \). Hence we have \( \mathcal{E} \subset \mathcal{B} \subset \emptyset \). It thus remains to show that \( \emptyset \subset \mathcal{O}(\mathcal{E}) \). Accordingly, let \( U \in \emptyset \setminus \{ \emptyset, X \} \) (it is clear that \( \emptyset, X \in \mathcal{O}(\mathcal{E}) \)). We want to show that \( U \in \mathcal{O}(\mathcal{E}) \). But since \( \mathcal{B} \) is a base for \( \emptyset \), there exists \( \mathcal{F} \subset \mathcal{B} \) such that \( U = \bigcup \mathcal{F} \).

If we can show that every \( F \in \mathcal{F} \) is a member of \( \mathcal{O}(\mathcal{E}) \), we’re done (since \( \mathcal{O}(\mathcal{E}) \) is closed under unions). But each \( F \in \mathcal{F} \) is a finite intersection of sets in \( \mathcal{E} \) (recall that \( \mathcal{E} \subset \mathcal{O}(\mathcal{E}) \) and that the latter, by being a topology, is closed under finite intersections). Hence \( \mathcal{F} \subset \mathcal{O}(\mathcal{E}) \), which concludes the proof of (c) and, with that, the proof. \( \square \)

### 3.2 Compactness

**3.13 Definition.** Let \( X \) be a topological space and let \( \mathcal{F} \) be a family of open sets of \( X \) with \( \bigcup \mathcal{F} = X \), then \( \mathcal{F} \) is called an open cover of \( X \).

**3.14 Definition.** Let \( X \) be a topological space. \( X \) is said to be compact if for every open cover \( \mathcal{F} \) of \( X \) there exists a finite \( \mathcal{E} \subset \mathcal{F} \) such that \( \mathcal{E} \) also is a cover of \( X \).

**Terminology.** The literature on the subject often refers to \( \mathcal{E} \) in Definition 3.14 as a finite subcover and defines a topological space \( X \) as compact if and only if every open cover of \( X \) has a finite subcover.

Although, the term ‘subcover’ might sometimes suggest itself to be ‘a cover of a subset of \( X \),’ rather than ‘a proper cover of \( X \) that is a subset of the original cover,’ it is always the latter meaning that’s in play.

There are other ways to describe compactness for topological spaces, that turns out to be equivalent to the description put forth in Definition 3.14. We will present two additional descriptions: one will follow just below – Proposition 3.16 – and the other will be presented as Theorem 4.17 on page 33.

**3.15 Definition.** Let \( X \) be any non-empty set. A family \( \mathcal{F} \subset \mathcal{P}(X) \) is said to have the finite intersection property if \( \bigcap \mathcal{E} \neq \emptyset \) for all finite \( \mathcal{E} \subset \mathcal{F} \).

**3.16 Proposition.** Let \( X \) be a topological space. Then \( X \) is compact if and only if for every family \( \mathcal{F} \) of closed sets with the finite intersection property, we have \( \bigcap \mathcal{F} \neq \emptyset \).

**Proof.** Starting with the only if-part. Assume that \( X \) is compact and let \( \mathcal{F} \) be a family of closed sets such that for all finite \( \mathcal{E} \subset \mathcal{F} \), we have \( \bigcap \mathcal{E} \neq \emptyset \).
We want to show that $\cap F \neq \emptyset$. But suppose this is not the case. Then we get

$$X \setminus \bigcap F = X \setminus \left( \bigcap_{F \in \mathcal{F}} F \right) = \bigcup_{F \in \mathcal{F}} (X \setminus F) = X,$$

where the third identity is due to Proposition 2.5.a. Notice that since all $F \in \mathcal{F}$ are closed, we have $X \setminus F$ open for all $F \in \mathcal{F}$. Thus we get $\{ X \setminus F : F \in \mathcal{F} \}$ to be an open cover of $X$ and since $X$ is compact this cover has a finite subcover, so there exists $\mathcal{E} \subset \mathcal{F}$ such that $\mathcal{E}$ is finite and $\bigcup_{F \in \mathcal{E}} (X \setminus F) = X$. But then – using Proposition 2.5.a again – we get

$$\bigcup_{F \in \mathcal{E}} (X \setminus F) = X \setminus \left( \bigcup_{F \in \mathcal{E}} F \right) = X \setminus \bigcap F = X,$$

rendering $\cap \mathcal{E} = \emptyset$ contradicting $\mathcal{F}$ having the finite intersection property. Hence we can’t have $\cap \mathcal{F} = \emptyset$. Hence $\cap \mathcal{F} \neq \emptyset$ follows, which concludes the only if-part of the proof.

To prove the if-part, suppose that whenever $\mathcal{F}$ is a family of closed sets with the finite intersection property, we have $\cap \mathcal{F} \neq \emptyset$.

We want to show that $X$ is compact in sense of Definition 3.14. Accordingly, let $\mathcal{F}$ be an open cover of $X$. We are done if we can show that there exists some finite $\mathcal{E} \subset \mathcal{F}$ such that $\bigcap \mathcal{E} = X$. But, by Proposition 2.5.b, we get

$$X \setminus \bigcup F = X \setminus \left( \bigcap_{F \in \mathcal{F}} F \right) = \bigcap_{F \in \mathcal{E}} (X \setminus F),$$

and since $\bigcup \mathcal{F} = X$, it follows that $\cap_{F \in \mathcal{F}} (X \setminus F) = \emptyset$. Also note that $\{ X \setminus F : F \in \mathcal{F} \}$ is a family of closed sets. Hence, by our initial assumption, $\{ X \setminus F : F \in \mathcal{F} \}$ can’t have the finite intersection property. This means that there exists a finite $\mathcal{E} \subset \mathcal{F}$ such that $\cap_{F \in \mathcal{E}} (X \setminus F) = \emptyset$. By Proposition 2.5.b, we have

$$\bigcap_{F \in \mathcal{F}} (X \setminus F) = X \setminus \left( \bigcup_{F \in \mathcal{E}} F \right) = X \setminus \bigcup \mathcal{E}.$$

Hence $\bigcup \mathcal{E} = X$, which concludes the if-part of the proof and with that the proof.

\[ \square \]

### 3.3 The product topology

#### 3.17 Definition. Let $\{X_\alpha\}_{\alpha \in A}$ be a non-empty family of topological spaces, and let $\{\pi_\alpha\}_{\alpha \in A}$ be the coordinate-mappings on $\prod_{\alpha \in A} X_\alpha$. Then the topology on $X = \prod_{\alpha \in A} X_\alpha$, generated by the set

$$\{ \pi_\alpha^{-1}(U_\alpha) : \alpha \in A \text{ and } U_\alpha \text{ is open in } X_\alpha \}$$

is called the product topology on $X$. 

24
At this point it is possible to formulate our main theorem.

3.18 Tychonoff’s theorem. Let \( \{X_{\alpha}\}_{\alpha \in A} \) be a non-empty family of compact topological spaces. Then \( X = \prod_{\alpha \in A} X_{\alpha} \) is compact with respect to the product topology.

We are not ready to give our proof of the theorem just yet. The proof we will give (starting on page 34) is based on the theory of nets (to be treated the next chapter).

The next proposition is key to our proof of Tychonoff’s theorem. Before stating it, however, we will find it convenient to introduce a new piece of terminology.

Terminology. If \( A \) is a set such that a statement \( S \) is true for all but a finite number of \( \alpha \in A \), then we will say that \( S \) is true for almost all \( \alpha \in A \).

3.19 Proposition. Let \( \{X_{\alpha}\}_{\alpha \in A} \) be a non-empty family of topological spaces. Then the set

\[ \mathcal{B} = \left\{ \prod_{\alpha \in A} U_{\alpha} : U_{\alpha} \text{ is open in } X_{\alpha} \text{ for all } \alpha \text{ and } U_{\alpha} = X_{\alpha} \text{ for almost all } \alpha \right\} \]

is a base for the product topology on \( X = \prod_{\alpha \in A} X_{\alpha} \).

Proof. Since the product topology on \( X \) is generated by the set

\[ \mathcal{E} = \{ \pi_{\alpha}^{-1}(U_{\alpha}) : \alpha \in A \text{ and } U_{\alpha} \text{ is open in } X_{\alpha} \} \]

where \( \{\pi_{\alpha}\} \) is the coordinate mappings on \( X \), we get – by Theorem 3.12 – that the set

\[ \mathcal{B}' = \left\{ \bigcap \mathcal{F} : \mathcal{F} \subset \mathcal{E} \text{ and } \mathcal{F} \text{ is finite} \right\} \cup \{X\} \]

is a base for \( X \). Moreover, since \( X \in \mathcal{E} \) (\( X = \pi_{\alpha}(X_{\alpha}) \) for any \( \alpha \in A \)) we get \( \bigcap \{X\} = X \in \mathcal{B}' \). Hence,

\[ \mathcal{B}' = \left\{ \bigcap \mathcal{F} : \mathcal{F} \subset \mathcal{E} \text{ and } \mathcal{F} \text{ is finite} \right\} \]

If we can show that \( \mathcal{B}' = \mathcal{B} \), we’re done.

Starting with \( \mathcal{B}' \subset \mathcal{B} \): Let \( U \in \mathcal{B}' \). Then there exists a finite number of elements \( \alpha_1, \ldots, \alpha_n \in A \) associated with sets \( U_{\alpha_j} \) open in the respective \( X_{\alpha_j} \), such that

\[ U = \bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}(U_{\alpha_j}) \].

25
Now, for each $\alpha \in A \setminus \{\alpha_1, \ldots, \alpha_n\}$, put $U_\alpha = X_\alpha$, and notice that we get $\prod_{\alpha \in A} U_\alpha$ to be a member of $B$, as well as
\[
\prod_{\alpha \in A} U_\alpha = \{ f \in X : f(\alpha_j) \in U_{\alpha_j} \text{ for all } j = 1, \ldots, n \}
= \bigcap_{j=1}^n \{ f \in X : \pi_{\alpha_j}(f) \in U_{\alpha_j} \}
= \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})
= U.
\]

Hence $U \in B$, so $B' \subset B$.

To prove $B \subset B'$, let $U \in B$. Then $U = \prod_{\alpha \in A} U_\alpha$ for some family $\{U_\alpha\}_{\alpha \in A}$ where $U_\alpha$ is open in $X_\alpha$ for all $\alpha$ and $U_\alpha = X_\alpha$ for almost all $\alpha$.

Put
\[
B = \{ \alpha \in A : U_\alpha \neq X_\alpha \}
\]
and notice that
\[
\bigcap_{\beta \in B} \pi_\beta^{-1}(U_\beta) \in B',
\]
since $B$ is finite, as well as
\[
\bigcap_{\beta \in B} \pi_\beta^{-1}(U_\beta) = \{ f \in X : f(\beta) \in U_\beta \text{ for all } \beta \in B \}
= \prod_{\alpha \in A} U_\alpha
= U.
\]

Hence, $U \in B'$, so $B \subset B'$ which concludes the proof. \[\square\]
Chapter 4

Nets

4.1 Directed sets

The theory of nets begin with the notion ‘directed set’. The class of directed sets is a subclass of the class of partially ordered sets where the former is separated from the latter as follows:

4.1 Definition. A partially ordered, non-empty set \((A, \preceq)\), is called a directed set if for any choice of \(\alpha, \beta \in A\), there exists \(\gamma \in A\) with \(\alpha \preceq \gamma\) and \(\beta \preceq \gamma\).

Terminology. If \((A, \preceq)\) is a directed set we also say that that \(A\) is directed by \(\preceq\). Whenever \(\preceq\) is understood or unspecified, we may drop reference to it and speak of \(A\), simply, as a directed set.

As with any partial ordering we may write ‘\(\beta \succeq \alpha\)’ for ‘\(\alpha \preceq \beta\)’.

4.2 Example. Let \(B\) be any non-empty set and put \(A = \{ A \subset B : A \text{ is finite} \}\). Note that for each pair of \(A, A' \in A\) we have \(A \cup A' \in A\) and \(A, A' \subset A \cup A'\). Hence \((A, \subset)\) is a directed set.

4.3 Proposition. Let \(X\) be a topological space and \(x\) a point in \(X\). Let \(N_x\) be the set of neighborhoods of \(x\).\(^1\) If we take ‘\(\preceq\)’ to designate the relation on \(N_x\) defined by:

\[ U \preceq V \text{ if and only if } U \supseteq V, \]

then \(N_x\) is directed by \(\preceq\).

Proof. We need to verify the following:

(a) \(\preceq\) is a partial ordering on \(N_x\),

\(^1\)\(N_x = \{ U \subset X : x \in U \}\).
(b) If \( U, V \in \mathcal{N}_x \) then there exists \( W \in \mathcal{N}_x \) such that \( U \preceq W \) and \( V \preceq W \).

Clearly \( \preceq \) is reflexive and anti-symmetric, and whenever we have \( U \preceq V \) and \( V \preceq W \) then \( U \supset V \supset W \), so \( U \supseteq W \) which is to say \( U \preceq W \). Hence \( \preceq \) is transitive which concludes (a).

To prove (b), note that whenever \( U, V \in \mathcal{N}_x \) then also \( x \in U^o \cap V^o \) which is open in \( X \) and thus a member of \( \mathcal{N}_x \), so we get both \( U \preceq U^o \cap V^o \) and \( V \preceq U^o \cap V^o \) (\( U \supset U^o \) and \( V \supset V^o \), by Definition 3.5.)

Following the literature on the subject, we will say that \( \mathcal{N}_x \) is directed by reverse inclusion.

**4.4 Proposition.** Let \((A, \preceq)\) and \((A', \preceq')\) be directed sets, and let \( \preceq'' \) be a relation with respect to \( A \times A' \) defined by:

\[(\alpha, \alpha') \preceq'' (\beta, \beta') \text{ if and only if } \alpha \preceq \beta \text{ and } \alpha' \preceq' \beta'.\]

Then \( A \times A' \) is directed by \( \preceq'' \).

**Proof.** Clearly \( \preceq'' \) is a partial-ordering on \( A \times A' \), so it remains to show that given \( (\alpha, \alpha'), (\beta, \beta') \in A \times A' \) there exists \( (\gamma, \gamma') \in A \times A' \) such that \( (\alpha, \alpha'), (\beta, \beta') \preceq'' (\gamma, \gamma') \). But note that there exists \( \gamma \in A \) with \( \alpha, \beta \preceq \gamma \) as well as a \( \gamma' \in A' \) with \( \alpha', \beta' \preceq' \gamma' \), so \( (\gamma, \gamma') \) works. \( \square \)

### 4.2 Nets and Subnets

**4.5 Definition.** Let \( X \) be any set and let \( A \) be a directed set. Any mapping in \( X^A \) is called a net in \( X \).

**Notation.** For a convenient way to denote a net of the form

\[A \to X : \alpha \mapsto x_\alpha\]

we will adopt the notation \( \langle x_\alpha \rangle_{\alpha \in A} \) or simply \( \langle x_\alpha \rangle \) (if \( A \) is understood). We will also say that nets of this form are based on \( A \).

**4.6 Example.** The set of positive integers is directed by the usual \( \leq \). Hence every sequence is a net. Indeed, consider an arbitrary sequence \( \{x_n\} \) in some set \( X \), and recall that this is in fact a short-hand notation for the mapping

\[\mathbb{Z}_+ \to X : n \mapsto x_n.\]
4.7 Example. Let $X$ be a non-empty set. Then any subset $M$ of $\mathcal{P}(X)$ with $X \in M$ is directed by $\subseteq$. Suppose that $M$ is a $\sigma$-algebra on $X$. Then every measure $\mu : M \to [0, \infty]$ is a net in $[0, \infty]$.

4.8 Definition. Let $(A, \preceq)$ and $(B, \preceq')$ be directed sets. A mapping $\beta \mapsto \alpha_\beta$ in $A \to B$ is called cofinal if for each $\alpha_0 \in A$ there is at least one $\beta_0 \in B$ such that whenever $\beta \preceq' \beta_0$ we have $\alpha_\beta \preceq \alpha_0$.

4.9 Example. Increasing mappings\(^2\) from some directed set $A$ into itself are always cofinal. Indeed, let $A$ be a set directed by $\preceq$ and let $\phi \in A^A$ be an increasing mapping (i.e. $\alpha \preceq \phi(\alpha)$ for all $\alpha \in A$). Then choose any $\alpha_0 \in A$. To be able to say that $\phi$ is cofinal we now need to find a $\beta_0 \in A$ such that whenever $\beta \preceq \beta_0$, we have $\phi(\beta) \geq \alpha_0$, but note that in this case we can simply choose $\alpha_0$ itself to work as such a $\beta_0$.

Staying with the case where $\phi \in A^A$ and $A$ being directed by $\preceq$, we’ve just seen (in Example 4.9) that if $\phi$ is increasing, then $\phi$ is cofinal. The converse, however, isn’t necessarily the case. In fact, there are cofinal mappings that map a directed set into itself, but aren’t increasing (cf. Example 4.10).

Likewise, it might also – under these circumstances – be tempting to make the conjecture that every cofinal mapping in $A^A$ is monotonically increasing, that is, whenever $\alpha \preceq \alpha'$ we have $\phi(\alpha) \preceq \phi(\alpha')$. But as the following example will illustrate, this is not the case either. Worth mentioning is also the fact that there are plenty of monotonically increasing mappings that are not cofinal, among which the mapping

$$x \mapsto \arctan(x)$$

in $\mathbb{R}^\mathbb{R}$ is one example. Indeed, simply pick $\alpha_0 = \frac{\pi}{2}$. There are no $\beta_0 \in \mathbb{R}$ such that $\arctan(\beta_0) \geq \frac{\pi}{2}$.

4.10 Example. Consider the mapping

$$\phi : \mathbb{Z}_+ \to \mathbb{Z}_+ : \alpha \mapsto (\alpha + 3) - (\alpha \mod 3) - (\alpha \mod 5).$$

First we will show that $\phi$ is cofinal. But notice that for any choice of $\alpha \in \mathbb{Z}_+$, we have $\phi(\alpha) \geq \alpha - 4$, or equivalently: $\phi(\alpha + 4) \geq \alpha$. Hence, if we pick an $\alpha_0 \in \mathbb{Z}_+$ we can simply put $\beta_0 = \alpha_0 + 4$ so that whenever we have $\beta \in \mathbb{Z}_+$ such that $\beta \geq \beta_0$,
we get

\[ \phi(\beta) = \phi(\beta + 4 - 4) \]
\[ \geq \beta - 4 \]
\[ \geq \beta_0 - 4 \]
\[ = (\alpha_0 + 4) - 4 \]
\[ = \alpha_0, \]

which is to say that \( \phi \) is cofinal.

As we run through the first 10 arguments of \( \phi \) we get the associated values according to the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(x) )</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>\cdots</td>
</tr>
</tbody>
</table>

This shows clearly that \( \phi \) is neither increasing (in the sense of \( \alpha \leq \phi(\alpha) \) for all \( \alpha \)) nor monotonically increasing (in the sense of having \( \phi(\alpha) \leq \phi(\alpha') \) whenever \( \alpha \leq \alpha' \)). This holds even when we cut off the domain of \( \phi \) to exclude all integers that aren’t sufficiently large. In fact, notice that for any \( \alpha \in \mathbb{Z}_+ \) of the form \( \alpha = 3n5m \) we have \( \phi(\alpha) = \alpha + 3 \), but since

\[ (3n5m + 1) \mod 3 > 0 \]

and

\[ (3n5m + 1) \mod 5 > 0, \]

we get

\[ \phi(\alpha + 1) \leq \alpha + 2 < \phi(\alpha). \]

Neither will \( \phi \) start to become increasing beyond any sufficiently large point. Indeed, suppose \( \alpha_0 \) were to be such a point. Then we cannot have any \( \alpha \in \mathbb{Z}_+ \) such that \( \alpha > \alpha_0 \) and \( \alpha \mod 5 = 4 \) (we would get \( \phi(\alpha) = \alpha - 1 - (\alpha \mod 3) < \alpha \)).

In preparation for the next definition (which is somewhat intricate), it is helpful to take note of the following:

Let \( \langle x_\alpha \rangle_{\alpha \in A} \) be a net in some set \( X \). Notice that if \( \phi \in A^B \) and \( B \) is any directed set besides \( A \), then the composite mapping

\[ \beta \mapsto \phi(\beta) \mapsto x_{\phi(\beta)} \]

from \( B \) to \( X \), is a net in \( X \) based on \( B \), which then may be denoted by \( \langle x_{\phi(\beta)} \rangle_{\beta \in B} \).
4.11 Definition. Let $A$ and $B$ be directed sets and $\langle x_\alpha \rangle_{\alpha \in A}$ a net in some set $X$. If $\phi$ is a cofinal mapping in $A^B$, then the net $\langle x_{\phi(\beta)} \rangle_{\beta \in B}$ is called a subnet of $\langle x_\alpha \rangle_{\alpha \in A}$.

4.12 Proposition. Let $s$, $s'$ and $s''$ be nets in the same set $X$. If $s''$ is a subnet of $s'$ and $s'$ is a subnet of $s$, then $s''$ is a subnet of $s$.

Proof. Assume $s$, $s'$ and $s''$ to be based on $(A, \preceq)$, $(B, \preceq')$ and $(C, \preceq'')$ respectively and take $\langle x_\alpha \rangle_{\alpha \in A}$ to denote $s$. That $s'$ is a subnet of $s$ is to say that there is a cofinal mapping $\phi \in A^B$ such that $s' = \langle x_{\phi(\beta)} \rangle_{\beta \in B}$.

And that $s''$ is a subnet of $s'$ is in equal fashion to say that there is a cofinal mapping $\psi \in B^C$ such that $s'' = \langle x_{\phi(\psi(\gamma))} \rangle_{\gamma \in C} = \langle x_{\phi(\psi(\gamma))} \rangle_{\gamma \in C}$.

Hence, if we can show that the composite mapping $\phi \circ \psi$ in $A^C$ is cofinal we're done. But let $\alpha_0 \in A$. We want to show that there exists $\gamma_0 \in C$ such that $\phi \circ \psi(\gamma) \triangleright \alpha_0$ for all $\gamma \triangleright'' \gamma_0$.

By $\phi$ being cofinal, we get $\beta_0 \in B$ such that $\phi(\beta) \triangleright \alpha_0$ for all $\beta \triangleright' \beta_0$, and by $\psi$ being cofinal, there exists $\gamma_0 \in C$ such that $\psi(\gamma) \triangleright' \beta_0$ for all $\gamma \triangleright'' \gamma_0$. It is thus clear that for all $\gamma \triangleright'' \gamma_0$, we have $\phi \circ \psi(\gamma) \triangleright \alpha_0$. Hence $\phi \circ \psi$ is cofinal, thus concluding the proof. \qed

4.13 Example. Let $\{x_n\}$ be a sequence in some non-empty set $X$. A subsequence, usually denoted by $\{x_{n_j}\}$ of $\{x_n\}$ is a composite mapping

$$\mathbb{N} \rightarrow \mathbb{N} \rightarrow \{x_n : n \in \mathbb{N}\} : j \mapsto n_j \mapsto x_{n_j}$$

such that $n_1 < n_2 < \ldots$

Recall that $\{x_n\}$ is a net in $X$ (cf Example 4.6). Moreover, by Example 4.9, the mapping $j \mapsto n_j$ is cofinal. Hence every subsequence is also a subnet.

Since every sequence $\{x_n\}$ qualifies as a net and every subsequence $\{x_{n_j}\}$ to $\{x_n\}$ qualifies as a subnet to $\{x_n\}$ (when the latter is viewed as a net), it is fair to say (to this extent at least) that the theory of nets and subnets is a generalization of the theory of sequences and subsequences. Many more analogies do exist and we will look at a few of them in the next section.
4.3 Nets and Compactness

It turns out that the notion of compactness in topological spaces can be described completely in terms of the theory of nets. This section gives the details.

4.14 Definition. Let \( \langle x_\alpha \rangle_{\alpha \in A} \) be a net in X and let \( U \subset X \). Then we say that:

(a) \( \langle x_\alpha \rangle \) is eventually in \( U \) if there exists \( \alpha_0 \in A \) with \( x_\alpha \in U \) for all \( \alpha \geq \alpha_0 \).

(b) \( \langle x_\alpha \rangle \) is frequently in \( U \) if for all \( \alpha \in A \), there exists \( \beta \geq \alpha \) with \( x_\beta \in U \).

The definition above puts no demands on the set \( X \). But note that the following definition only deals with cases in which the set \( X \) is a topological space.

4.15 Definition. Let \( X \) be a topological space and let \( \langle x_\alpha \rangle_{\alpha \in A} \) be a net in \( X \). Then we say that:

(a) \( \langle x_\alpha \rangle \) converge to a point \( x \in X \) if for all neighborhoods \( U \) of \( x \), \( \langle x_\alpha \rangle \) is eventually in \( U \).

(b) A point \( x \in X \) is a cluster point of \( \langle x_\alpha \rangle \) if for all neighborhoods \( U \) of \( x \), \( \langle x_\alpha \rangle \) is frequently in \( U \).

4.16 Proposition. Let \( \langle x_\alpha \rangle_{\alpha \in A} \) be a net in a topological space \( X \) and let \( x \in X \). Then \( x \) is a cluster-point of \( \langle x_\alpha \rangle \) if and only if \( \langle x_\alpha \rangle \) has a subnet converging to \( x \).

Proof. Starting with the if-part: Assume that \( \langle x_{\alpha_0} \rangle_{B \in B} \) is a subnet converging to \( x \) (we may take \( \leq \) and \( \leq' \) to designate the relations that directs \( A \) and \( B \) respectively).

We need to show that for any neighborhood \( U \) of \( x \), \( \langle x_\alpha \rangle \) is frequently in \( U \).

Hence let \( U \) be an arbitrary neighborhood of \( x \) and let \( \alpha_0 \in A \). If we can find an \( \alpha \geq \alpha_0 \) with \( x_\alpha \in U \), then the if-part follows. But \( \beta \mapsto \alpha_0 \) being cofinal, gives us a \( \beta_1 \in B \) with \( \alpha_0 \geq \beta_1 \) for all \( \beta \geq \beta_1 \), and \( \langle x_{\alpha_0} \rangle \) converging to \( x \) gives us a \( \beta_2 \in B \) with \( x_{\alpha_0} \in U \) for all \( \beta \geq \beta_2 \). Now, since \( B \) is directed by \( \leq' \) we get a \( \beta \in B \) with \( \beta_1 \leq \beta \). Note that \( \alpha_0 \geq \alpha_0 \) and \( x_{\alpha_0} \in U \).

It remains to show the only if-part. To do this assume that \( x \) is a cluster-point of \( \langle x_\alpha \rangle \).

Our aim is to find a subnet converging to \( x \).

Let \( N \) be the set of neighborhoods of \( x \) in \( X \). By Proposition 4.3, \( N \) is directed by reverse inclusion. Hence, by Proposition 4.4, \( N \times A \) is directed by \( \leq'' \), defined by:

\[ (U, \gamma) \leq'' (U', \gamma') \quad \text{if and only if} \quad U \supset U' \quad \text{and} \quad \gamma \leq \gamma'. \]

Next, for all \( (U, \gamma) \in N \times A \) we put

\[ X_{(U, \gamma)} = \{ \alpha \in A : x_\alpha \in U \quad \text{and} \quad \alpha \geq \gamma \}. \]
Notice that \( x \) being cluster-point of \( \langle x_\alpha \rangle \), together with \( U \) being a neighborhood of \( x \), renders each \( X_{(U,\gamma)} \) non-empty. Indeed, given some \( (U,\gamma) \in N \times A \), we get \( \langle x_\alpha \rangle \) frequently in \( U \), thus there exists \( \alpha \succeq \gamma \) with \( x_\alpha \in U \). This \( \alpha \) is an element of \( X_{(U,\gamma)} \).

Now by Proposition 2.9 we get

\[
\prod_{(U,\gamma) \in N \times A} X_{(U,\gamma)} \neq \emptyset.
\]

Hence there exists a mapping \( (U,\gamma) \mapsto \alpha_{(U,\gamma)} \) in \( A^{N \times A} \) such that \( \alpha_{(U,\gamma)} \in X_{(U,\gamma)} \) for all \( (U,\gamma) \in N \times A \).

This mapping is cofinal. Indeed, letting \( \alpha_0 \in A \) we may simply choose any \( V_0 \in N \) and consider elements \( (U,\gamma) \succeq \gamma \) \( (V_0,\alpha_0) \) to see directly that \( \alpha_{(U,\gamma)} \succeq \gamma \succeq \alpha_0 \).

We thus get \( \langle x_{\alpha_{(U,\gamma)}} \rangle \) to be a subnet of \( \langle x_\alpha \rangle \).

To conclude the proof it remains only to show that \( \langle x_{\alpha_{(U,\gamma)}} \rangle \) converge to \( x \). Accordingly, let \( V_0 \) be a neighborhood of \( x \). We need to find some \( (U_0,\gamma_0) \in N \times A \) such that whenever \( (U,\gamma) \succeq \gamma \) \( (U_0,\gamma_0) \) we have \( x_{\alpha_{(U,\gamma)}} \in V_0 \). But \( (V_0,\gamma_0) \) works for any choice of \( \gamma_0 \in A \). Indeed, let \( (U,\gamma) \succeq \gamma \) \( (V_0,\gamma_0) \). Then it follows immediately that \( x_{\alpha_{(U,\gamma)}} \in U \subset V_0 \).

\[\Box\]

4.17 Theorem. Let \( X \) be a topological space. Then \( X \) is compact if and only if every net in \( X \) has a convergent subnet.

Proof. Starting with the only if-part: Assume \( X \) is compact and let \( \langle x_\alpha \rangle \) be a net, based on \( (A,\succeq) \), in \( X \). We need to show that \( \langle x_\alpha \rangle \) has a subnet that converges to some \( x \in X \). Start by putting, for all \( \alpha \in A \)

\[
E_\alpha = \{ x_\beta : \beta \succeq \alpha \}.
\]

Note that the family \( \{E_\alpha\}_{\alpha \in A} \) has the finite intersection property (described in Definition 3.15). \(^3\)

Since for all \( \alpha \), \( E_\alpha \) is a subset of its closure \( \overline{E_\alpha} \), the family \( \{\overline{E_\alpha}\}_{\alpha \in A} \) also has the finite intersection property, and since \( X \) is compact it follows from Proposition 3.16 that \( \bigcap_{\alpha \in A} \overline{E_\alpha} \) is non-empty, so there exists \( x \in \bigcap_{\alpha \in A} \overline{E_\alpha} \).

We will now show that this \( x \) is a cluster-point of \( \langle x_\alpha \rangle_{\alpha \in A} \), which by Proposition 4.16, would guarantee the existence of a subnet of \( \langle x_\alpha \rangle \), that converge to \( x \). But

\(^3\)This follows from the induction-principle. Indeed, if \( \bigcap_{\gamma \in \mathcal{B}} E_\gamma \neq \emptyset \) (trivially true whenever \( |\mathcal{B}| = 1 \)) for all subsets \( \mathcal{B} \) of \( A \) of size \( n \), then for any subset \( \mathcal{B} \) of \( A \) of size \( n+1 \), we can yank some \( \beta_0 \) from \( \mathcal{B} \) and get \( \bigcap_{\gamma \in \mathcal{B} \setminus \{\beta_0\}} E_\gamma \neq \emptyset \), because \( \mathcal{B} \setminus \{\beta_0\} \) is of size \( n \). Thus we get some \( \beta_1 \in A \) with \( x_{\beta_0} \in E_{\gamma} \) for all \( \gamma \in B \setminus \{\beta_0\} \), which means that \( \beta_1 \succeq \gamma \) for all \( \gamma \in B \setminus \{\beta_0\} \). Now by Definition 4.1 there exists \( \beta_2 \in A \) with \( \beta_0, \beta_1 \prec \beta_2 \), thus \( x_{\beta_2} \in E_{\gamma} \) for all \( \gamma \in B \), rendering \( \bigcap_{\gamma \in \mathcal{B}} E_\gamma \) non-empty.
let $U$ be a neighborhood of $x$ in $X$ and let $\alpha_0 \in A$, then we have $x \in \overline{E_{\alpha_0}}$, which, in particular, renders $U \cap E_{\alpha_0} \neq \emptyset$. \footnote{If $x \in E_{\alpha_0}$ then $x$ itself is an element in $U \cap E_{\alpha_0}$. Else, if $x \notin E_{\alpha_0}$ then we have $x \in \text{acc}(E_{\alpha_0})$, since by Proposition 3.8, $\overline{E_{\alpha_0}} = E_{\alpha_0} \cup \text{acc}(E_{\alpha_0})$, so $U \cap E_{\alpha_0} \neq \emptyset$ follows.}

Hence there exists $\beta \geq \alpha_0$ with $x_\beta \in U$, so $x$ is a cluster-point of $\langle x_\alpha \rangle$, which concludes the only if-part of the proof.

To prove the if-part: assume that every net in $X$ has a convergent subnet. We want to prove that $X$ is compact.

First note, however, that by Proposition 4.16, we have just assumed that every net in $X$ has at least one cluster-point. Now, suppose $X$ is not compact. Our plan is to prove this to be impossible by using this assumption to find a net in $X$ that doesn’t have any cluster-points.

Since $X$ is not compact we get an open cover $\{U_\beta\}_{\beta \in B}$ of $X$ that does not have any finite sub-covers. Put

$$A = \{ A \subset B : A \text{ is finite} \}.$$ 

Note that $(A, \subset)$ is a directed set (cf. Example 4.2) and for each $A \in A$, $X \setminus \bigcup_{\alpha \in A} U_\alpha$ is non-empty. Hence by Proposition 2.9 we have

$$\prod_{A \in A} \left( X \setminus \bigcup_{\alpha \in A} U_\alpha \right) \neq \emptyset.$$ 

Thus we get a mapping $A \mapsto x_A$ in $X^A$ such that $x_A \in X \setminus \bigcup_{\alpha \in A} U_\alpha$ for all $A \in A$. This mapping is clearly a net in $X$. Denote it by $\langle x_A \rangle$.

Now we claim that $\langle x_A \rangle$ does not have any cluster-points. Indeed, let $x$ be any element of $X$. Since $\{U_\beta\}_{\beta \in B}$ is an open cover of $X$ we get $x \in U_\beta$ for some $\beta \in B$. In particular $U_\beta$ is a neighborhood of $x$. Note that $\{\beta\} \in A$ and that for all $A \supset \{\beta\}$ we have

$$x_A \notin \bigcup_{\alpha \in A} U_\alpha \supset U_\beta,$$

that is, $x_A \notin U_\beta$ for all $A \supset \{\beta\}$, which is to say that $\langle x_A \rangle_{A \in A}$ is not frequently in $U_\beta$, so $x$ is not a cluster-point of $\langle x_A \rangle$, which concludes the if-part of the proof and with that the proof.

$\square$

### 4.3.1 A proof of Tychonoff’s theorem

Since our proof of Tychonoff’s theorem is quite involved, we will – for readability-purposes – make use of a full section to outline it.
Throughout this section, let \( \{X_\alpha\}_{\alpha \in \mathcal{A}} \) be a non-empty family of compact topological spaces. To prove Tychonoff’s theorem (page 25), which is the statement that \( \prod_{\alpha \in \mathcal{A}} X_\alpha \) is compact with respect to the product topology, it is – by Theorem 4.17 – sufficient to show that every net in \( \prod_{\alpha \in \mathcal{A}} X_\alpha \) has a cluster point. Accordingly, let \( \langle f_i \rangle_{i \in \mathcal{I}} \) be a net in \( \prod_{\alpha \in \mathcal{A}} X_\alpha \).

Our aim is to show that \( \langle f_i \rangle_{i \in \mathcal{I}} \) has a cluster point.

To begin with, note that for each subset \( \mathcal{B} \) of \( \mathcal{A} \), the restrictions \( f_i|_{\mathcal{B}} \) of the mappings \( f_i \) are all members of the product \( \prod_{\alpha \in \mathcal{B}} X_\alpha \). Hence \( \langle f_i|_{\mathcal{B}} \rangle_{i \in \mathcal{I}} \) is a net in \( \prod_{\alpha \in \mathcal{B}} X_\alpha \).\(^5\) Sets of this form may be called subproducts of \( \prod_{\alpha \in \mathcal{A}} X_\alpha \) and they come with their respective product topology. With this in mind put

\[
\mathcal{P} = \bigcup_{\mathcal{B} \in \mathcal{P}(\mathcal{A})} \left\{ p \in \prod_{\alpha \in \mathcal{B}} X_\alpha : p \text{ is a cluster point of } \langle f_i|_{\mathcal{B}} \rangle_{i \in \mathcal{I}} \right\},
\]

and define a relation \( \leq \) on \( \mathcal{P} \) as:

\[
p \leq q \text{ if and only if } q \text{ is an extension of } p,
\]

with the meaning of ‘extension’ taken in the sense of Definition 2.1.

Notice that \( (\mathcal{P}, \leq) \) is a partially ordered set. Another way to view \( \mathcal{P} \) is as a subset to \( \mathcal{A} \times \left( \bigcup_{\alpha \in \mathcal{A}} X_\alpha \right) \). In fact the ordering \( \leq \) is nothing but the standard subset ordering \( \subset \) on \( \mathcal{A} \times \left( \bigcup_{\alpha \in \mathcal{A}} X_\alpha \right) \), only restricted to \( \mathcal{P} \), which in particular means that if \( p \) and \( q \) are members of \( \mathcal{P} \) and \( B_p \) and \( B_q \) are their respective domains, then to say that \( p \leq q \), is simply to say that we have \( B_p \subset B_q \) and that \( q(\alpha) = p(\alpha) \) for all \( \alpha \in B_p \).

Our plan is to show that \( \mathcal{P} \) is non-empty and inductively ordered by \( \leq \) and thus, by Zorn’s lemma, we will get a maximal element \( p^* \in \mathcal{P} \). If the domain of \( p^* \) turns out to be \( \mathcal{A} \) (which it will,) then we’re done.

**4.18 Lemma.** Let \( \gamma \in \mathcal{A} \). Then the net \( \langle f_i|_{\{\gamma\}} \rangle_{i \in \mathcal{I}} \) has a cluster point in \( \prod_{\alpha \in \{\gamma\}} X_\alpha \).

**Proof.** Since \( X_\gamma \) is compact the net \( \langle f_i(\gamma) \rangle \) in \( X_\gamma \), has – by Theorem 4.17 – a cluster point \( p \) in \( X_\gamma \).

Notice that

\[
\prod_{\alpha \in \{\gamma\}} X_\alpha = X_\gamma^{\{\gamma\}} = \{(\gamma, x) : x \in X_\gamma \}.
\]

We will proceed to show that \( \{(\gamma, p)\} \) is a cluster point of \( \langle f_i|_{\{\gamma\}} \rangle_{i \in \mathcal{I}} \). Accordingly, let \( U \subset X_\gamma^{\{\gamma\}} \) be neighborhood of \( \{(\gamma, p)\} \). We need to show that \( \langle f_i|_{\{\gamma\}} \rangle \) is frequently in \( U \). But by Proposition 3.19, the set

\[
\left\{ \prod_{\alpha \in \{\gamma\}} U_\alpha : U_\alpha \text{ is open in } X_\alpha \right\} = \{ U_\gamma^{\{\gamma\}} : U_\gamma \text{ is open in } X_\gamma \}
\]

\(^5\)Chernoff [1], denotes this net by \( \{ f_i : i \in I \} \).
is a base for the product topology on $X^\gamma$. Thus there exists an open set $U_\gamma$ in $X_\gamma$
such that $\{(\gamma, p)\} \in U^\gamma \subset U$.

Now clearly, $\langle f_i(\gamma) \rangle$ is frequently in $U_\gamma$. Hence, $\langle f_\beta(\gamma) \rangle$ is frequently in $U^\gamma$. Indeed, $f_\beta(\gamma) = \{f_i(\gamma)\}$ for all $i \in I$ and $U^\gamma = \{(\gamma, x) : x \in U_\gamma\}$, so we have $f_\beta(\gamma) \in U^\gamma$ whenever $f_\beta(\gamma) \in U_\gamma$, which concludes the proof. □

In particular, Lemma 4.18 shows that $\mathcal{P}$ is non-empty. Indeed, there exists a mapping $\{\gamma \to X_\gamma\}$ that is a member of $\mathcal{P}$ (namely the cluster point to $\langle f_\beta(\gamma)\rangle_{\beta \in I}$) for all $\gamma \in A$. Next, we will turn to show that $\mathcal{P}$ is inductively ordered by $\leq$.

4.19 Lemma. Let $\mathcal{T}$ be a non-empty, totally ordered subset of $(\mathcal{P}, \leq)$. Then there exists $p^* \in \mathcal{P}$ such that $p^*$ is an upper bound for $\mathcal{T}$.

Proof. Put

$$p^* = \bigcup \mathcal{T},$$

and for each $p \in \mathcal{T}$, denote the domain of $p$ by $B_p$. We will show that $p^*$ is an upper bound for $\mathcal{T}$ in $\mathcal{P}$.

We’re done if we can show the following three items:

(a) $p^* \in \prod_{\alpha \in B^*} X_\alpha$, where $B^* = \bigcup_{p \in \mathcal{T}} B_p$,

(b) $p^*$ is an extension of every $p \in \mathcal{T}$,

(c) $p^*$ is a cluster point of the net $\langle f\rangle_{B^*}$.

Indeed, if (a) and (c) holds, then $p^* \in \mathcal{P}$, and if in addition (b) holds, then $p^*$ is also an upper bound for $\mathcal{T}$.

Starting with (a): Clearly $p^* \subset B^* \times (\bigcup_{\alpha \in B^*} X_\alpha)$, thus – for (a) – it only remains to show that for every $\alpha \in B^*$, two conditions holds, namely

(i) there exists $x \in X_\alpha$ such that $(\alpha, x) \in p^*$,

(ii) if $x' \in \bigcup_{\alpha \in B^*} X_\alpha$ with $(\alpha, x') \in p^*$, then $x = x'$.

But let $\alpha \in B^*$. By the fact that $\alpha \in B_p$ for some $p \in \mathcal{T}$, we get $(\alpha, p(\alpha)) \in p \subset p^*$, which is (i). Now suppose $x \in \bigcup_{\alpha \in B^*} X_\alpha$ such that $(\alpha, x) \in p^*$. Then there exists $q \in \mathcal{T}$ with $x = q(\alpha)$ and since $\mathcal{T}$ is totally ordered by extension, we have either $p$ being an extension of $q$ or $q$ being an extension of $p$. Either way, $p(\alpha) = q(\alpha)$ holds. Hence (ii), and (a) is clear.

Note that (b) now follows immediately. In fact, if $p \in \mathcal{T}$ and $\alpha \in B_p$ then $(\alpha, p(\alpha)) \in p \subset p^*$, and by $p^*$ being a mapping, there can be no $x$ other that $p(\alpha)$ such that $(\alpha, x) \in p^*$, which is to say that $p(\alpha) = p^*(\alpha)$, so $p^*$ is an extension of $p$. 36
Thus it only remains to prove (c).

Accordingly, let $U$ be a neighborhood of $p^*$ in $\prod_{\alpha \in B^*} X_\alpha$. We need to show that $\langle f_i|B^* \rangle$ is frequently in $U$. But by Proposition 3.19, we get $\prod_{\alpha \in B^*} U_\alpha = X_\alpha$ for almost all $\alpha$. Hence, there exists a family $\{U_\alpha\}_{\alpha \in B^*}$ such that $p^* \in \prod_{\alpha \in B^*} U_\alpha \subseteq U$, all $U_\alpha$ are open in the respective $X_\alpha$ and $U_\alpha = X_\alpha$ for almost every $\alpha \in B^*$. Thus, we’re done if we can show that the net $\langle f_i|B^* \rangle$ is frequently in $\prod_{\alpha \in B^*} U_\alpha$.

Accordingly, let $i_0 \in I$. We need to find some $i \geq i_0$ with $f_i|B^* \in \prod_{\alpha \in B^*} U_\alpha$. But, let $\alpha_1, \ldots, \alpha_n$ be those $\alpha \in B^*$ for which $U_\alpha \neq X_\alpha$ (they’re finite in number) and note that if we can find some $i \geq i_0$ with $f_i(\alpha_j) \in U_{\alpha_j}$ for all $j = 1, \ldots, n$, we’re done. Indeed, for $\alpha \in B^* \setminus \{\alpha_1, \ldots, \alpha_n\}$, we have $f_i(\alpha) \in U_\alpha$ trivially, since then $U_\alpha = X_\alpha$. But the family of domains $\{B_p\}_{p \in \mathcal{J}}$ for the members of $\mathcal{J}$, is totally ordered by $\subseteq$. Hence there exists a $p \in \mathcal{J}$ with $\{\alpha_1, \ldots, \alpha_n\} \subseteq B_p$. Clearly $p \in \prod_{\alpha \in B_p} U_\alpha$, since otherwise $p^*$ wouldn’t extend $p$ (which it does).

Again by Proposition 3.19, we get $\prod_{\alpha \in B_p} U_\alpha$ to be a neighborhood of $p$, and since $p$ is a cluster point to the net $\langle f_i|B_p \rangle_{i \in I}$, this net is frequently in $\prod_{\alpha \in B_p} U_\alpha$. Thus there exists an $i \geq i_0$ with $f_i|B_p \in \prod_{\alpha \in B_p} U_\alpha$. This, in particular, is to say that $f_i(\alpha_j) \in U_{\alpha_j}$ for all $j = 1, \ldots, n$, which concludes the proof. \□

Recall that if we can show that $\mathcal{P}$ has a member whose domain is $A$, Tychoff’s theorem will follow. But by Lemma 4.19, $\mathcal{P}$ is inductively ordered. And since $\mathcal{P}$ is also non-empty, it follows by Zorn’s lemma that $\mathcal{P}$ has a maximal element $p^*$. Denote the domain of $p^*$ by $B^*$.

4.20 Lemma. $B^* = A$

Proof. Suppose that $B^* \neq A$. Then there exists $\gamma \in A \setminus B^*$. Since $p^*$ is a cluster point of $\langle f_i|B^* \rangle$, it follows from Proposition 4.16 that this net has a subnet

$$\langle f_{i(j)}|B^* \rangle_{j \in J}$$

converging to $p^*$. Recall that $J$ is some directed set and $j \mapsto i(j)$ is some cofinal mapping in $I^J$.

Now, consider the net

$$\langle f_{i(j)}|\gamma \rangle_{j \in J} \quad (4.2)$$

\^\text{\textsuperscript{6}In [1] Chernoff refers to such a set as a \textit{basic neighborhood} of $p^*$.}
in $\prod_{\alpha \in \gamma} X_\alpha$. By Lemma 4.18, there exists $p_\gamma \in \prod_{\alpha \in \gamma} X_\alpha$ such that $p_\gamma$ is a cluster point to the net (4.2). Hence, by Proposition 4.16, the net (4.2) has a subnet that converges to $p_\gamma$. Notice how this subnet must be of the form:

$$\langle f_{i(j(k))}(\gamma) \rangle_{k \in K} \quad (4.3)$$

where $K$ is some directed set and the mapping $k \mapsto j(k)$ is some cofinal mapping in $J^K$.

Put $q = p^* \cup p_\gamma$. Clearly $q \in \prod_{\alpha \in B^* \cup \gamma} X_\alpha$. Moreover, we have $q > p^*$. Thus we can’t have $q \in \mathcal{P}$, since this contradicts the maximality of $p^*$. Hence, if we can show that $q \in \mathcal{P}$, using the assumption $B^* \neq A$, we’re done.

But notice that, by Proposition 4.12, the net

$$\langle f_i|B^* \cup \{\gamma\} \rangle_{i \in I} \quad (4.4)$$

in $\prod_{\alpha \in B^* \cup \gamma} X_\alpha$ is a subnet of the net

$$\langle f_i|B^* \cup \{\gamma\} \rangle_{i \in I} \quad (4.5)$$

Hence, if the net (4.4) converges to $q$, then by Proposition 4.16, $q$ is a cluster point of the net (4.5), which is to say that $q \in \mathcal{P}$.

Before proceeding to show this it is worthwhile to take note of the fact that there are three different directed sets in play here, that doesn’t necessarily have anything to do with each other, other than that they furnish a foundation for a pair of cofinal mappings to connect them according to

$$K \rightarrow J \rightarrow I.$$ 

The point is that they cannot be assumed to be directed by the same ordering (as if they all where subsets of, say $\mathbb{N}$.) Hence we can but assume $I, J, K$ to be directed by $\leq, \leq', \leq''$, respectively.

To show that the net (4.4) converges to $q$, let $U$ be a neighborhood of $q$ in $\prod_{\alpha \in B^* \cup \gamma} X_\alpha$. We need to find $k_0 \in K$ such that whenever $k \geq k_0$, we have $f_{i(j(k))}|B^* \cup \{\gamma\} \in U$.

But by Proposition 3.19, $U$ has at least one subset of the form

$$\prod_{\alpha \in B^* \cup \gamma} U_\alpha \quad (4.6)$$

that contains $q$ and where $U_\alpha$ is open in $X_\alpha$ for all $\alpha$ and $U_\alpha = X_\alpha$ for almost all $\alpha$. Clearly $p^* \in \prod_{\alpha \in B^*} U_\alpha$ and $p_\gamma \in \prod_{\alpha \in \gamma} U_\alpha$, since $q$ is an extension of both
$p^*$ and $p_\gamma$. Hence by Proposition 3.19, these sets are neighborhoods of $p^*$ and $p_\gamma$ respectively.

Since the net (4.1) converges to $p^*$ there exists $j_0 \in J$ such that whenever $j \succ j_0$, we have

$$f_{i(j)} B^* \in \prod_{\alpha \in B^*} U_\alpha.$$  

In addition: since the mapping $k \mapsto j(k)$ is cofinal, there exists $k_1 \in K$ such that $j(k) \succ j_0$ for all $k \succ k_1$.

Since the net (4.3) converges to $p_\gamma$, it is eventually in

$$\prod_{\alpha \in \{\gamma\}} U_\alpha$$

so there exists $k_2 \in K$ such that whenever $k \succ k_2$ we have,

$$f_{i(j(k))} \{\gamma\} \in \prod_{\alpha \in \{\gamma\}} U_\alpha.$$  

Now, since $(K, \preceq''')$ is a directed set, there exists $k_0 \in K$ with $k_1, k_2 \preceq''' k_0$. At this point we can verify directly that whenever $k \succ''' k_0$ we have

$$f_{i(j(k))} B^* \cup \{\gamma\} \in \prod_{\alpha \in B^* \cup \{\gamma\}} U_\alpha$$

that as indicated by (4.6), is a subset of $U$, thus concluding the proof.  

\[\square\]

Tychonoff’s theorem follows.
Appendix A

A proof of Zorn’s lemma

A.1 The Bourbaki fixed-point theorem

A.1 Definition. Let \((A, \leq)\) be a partially ordered set. A mapping \(f \in A^A\) is said to be increasing, if \(x \leq f(x)\) for all \(x \in A\).

We remind the reader at this point that a set \(S\) is said to be strictly inductively ordered if whenever \(T\) is a non-empty, totally ordered subset of \(S\), then \(T\) has a least upper bound in \(S\).

Notice that the only difference between a set \(S\) being strictly inductively ordered and inductively ordered, is that the upper bounds for the non-empty, totally ordered subsets are least upper bounds in the former.

A.2 Bourbaki fixed-point theorem. \(^1\) Let \(A\) be a non-empty, strictly inductively ordered set and let \(f\) be an increasing mapping in \(A^A\). Then there exists \(x_0 \in A\) such that \(f(x_0) = x_0\).

It is fair to say that the proof is rather technical in the sense that it is quite long and, for the sake of readability, has to be broken down into several smaller steps. For these reasons we will let it occupy its own section.

A.1.1 A proof of the Bourbaki fixed-point theorem

Throughout this section let \(A\) be a set strictly inductively ordered by \(\leq\) and let \(f \in A^A\) be an increasing mapping on \(A\). If we can show that \(f\) has a fixed point, the

\(^1\)This is essentially the same theorem that is stated in [5, page 881]. There Lang credits the Bourbaki group (of which he was a member himself.) The naming here, however, is due to the present author and is not generally recognized. A fixed point of a mapping in \(A^A\) is, of course, a point that is mapped to itself.
Bourbaki fixed-point theorem will follow. We shall conclude this through a series of lemmas.

Before proceeding, fix some \( a \in A \) (whose existence is guaranteed by \( A \) being non-empty) and let \( A \) designate the set of all \( B \subset A \) such that:

(i) \( a \in B \),

(ii) \( f(B) \subset B \),

(iii) For all non-empty, totally ordered subsets \( T \) of \( B \), the least upper bound for \( T \) is a member of \( B \).

The point here is that each totally ordered member of \( A \) will contain a fixed point of \( f \). Hence our task can be translated into showing that \( A \) contains such an element. To be precise:

**A.3 Lemma.** If \( M \in A \) and \( M \) is totally ordered, then there exists \( b \in M \) with \( f(b) = b \).

*Proof.* Clearly \( M \) is non-empty. So by \( A \) being strictly inductively ordered, we get some \( b \in A \) such that \( b \) is the least upper bound for \( M \) in \( A \). Moreover \( b \in M \) (\( M \) is a totally ordered subset of itself!) Hence \( f(b) \in M \), which renders \( f(b) \leq b \). But since \( f \) is increasing, we still have \( b \leq f(b) \), so \( b \leq f(b) \leq b \) which is to say \( f(b) = b \).

\( \Box \)

It will be made clear that the set \( \bigcap A \), is such an \( M \) as mentioned in Lemma A.3.

Through remainder of this section, take \( M \) to designate \( \bigcap A \).

**A.4 Lemma.** \( M \in A \)

*Proof.* That \( M \) is a subset of \( A \) follows immediately from the identity \( M = \bigcap A \). Hence it remains to show that \( M \) satisfies each of the three terms given in the definition of \( A \).

First we need to show that \( a \in M \), but \( a \in B \) for all \( B \in A \), which is to say that \( a \in \bigcap A \), or equally \( a \in M \).

Next we need to show that \( f(M) \subset M \). Accordingly, let \( x \in M \). Since \( x \in B \), for every \( B \in A \), it follows that \( f(x) \in B \) for those \( B \) (that is, \( f(x) \in \bigcap A \)). Hence \( f(M) \subset M \).

Finally, let \( T \) be a non-empty, totally ordered subset of \( M \) and let \( b \) be the least upper bound for \( T \) in \( A \). We need to show that \( b \in M \), but for all \( B \in A \) we have \( T \subset M \subset B \), so \( b \in B \) for every \( B \in A \), which is to say that \( b \in \bigcap A = M \), which conclude the proof.

\( \Box \)
Thus it remains to show that $M$ is totally ordered. This will be done through Lemma A.5–A.8.

**A.5 Lemma.** For all $x \in M$, $a \leq x$.

*Proof.* Since $M$ is a subset of every member of $\mathcal{A}$ it suffices to show that the set

$$A' = \{ x \in A : a \leq x \}$$

is a member of $\mathcal{A}$. But clearly $a \in A'$, and if $x \in A'$ then $a \leq x \leq f(x)$ (since $f$ is increasing,) so $f(A') \subset A'$.

Finally, if $T$ is a non-empty, totally ordered subset of $A'$ and $b$ the least upper bound for $T$ in $A$, then $x \leq b$ for some $x \in T$ (since $T$ is non-empty) and thus, by $a \leq x$ (since $T \subset A'$,) we get $a \leq b$, which renders $b \in A'$, concluding that $A' \in \mathcal{A}$. $\square$

We now introduce the last two pieces of notation used exclusively for technical purposes within this section.

For each $c \in M$ put

$$M_c = \{ x \in M : x \leq c \text{ or } f(c) \leq x \}.$$

If $c \in M$ such that for all $x \in M$ with $x < c$ we have $f(x) \leq c$, then $c$ is called an *extreme point* of $M$.

**A.6 Lemma.** If $c$ is an extreme point of $M$ then $M_c = M$.

*Proof.* Let $c$ be an extreme point of $M$. Note that it is sufficient to prove that $M_c \in \mathcal{A}$.

By Lemma A.5, we get $a \leq c$ (since $c \in M$), and thus $a \in M_c$, so $M_c$ satisfies criterion (i) for being a member of $\mathcal{A}$.

Next, we want to show that (ii) holds for $M_c$, namely that $f(M_c) \subset M_c$. Accordingly let $x \in M_c$. Then one of $x < c$ or $x = c$ or $f(c) \leq x$, must hold. And by $x \leq f(x)$, the third case translates to $f(c) \leq f(x)$ and here $f(x) \in M_c$ follows immediately.

If $x < c$ then, by $c$ being an extreme point of $M$, we get $f(x) \leq c$ and thus $f(x) \in M_c$. If $x = c$ then, of course, $f(c) = f(x)$ rendering $f(x) \in M_c$. Hence $f(M_c) \subset M_c$.

Finally, to show that (iii) holds for $M_c$, let $T$ be a non-empty, totally ordered subset of $M_c$ and $b$ the least upper bound for $T$ in $A$. We need to prove that $b \in M_c$. Note that if there exists $x \in T$ with $f(c) \leq x$, then, since $x \leq b$, by $b$ being an upper bound for $T$ in $A$, $f(c) \leq b$ follows and thus $b \in M_c$. We may thus assume that there are no such $x$ in $T$, leaving us with $x \leq c$ for all $x \in T$, which renders $c$ to be
an upper bound for $T$ in $A$. Hence $b \leq c$, since $b$ is the least upper bound for $T$.
Thus we get $b \in M_c$, concluding that $M_c \in A$ and thereby $M \subset M_c$ followed by $M = M_c$. □

A.7 Lemma. Each element of $M$ is an extreme point of $M$.

Proof. Let $E$ designate the set of extreme points of $M$. Since $E \subset M$ it is sufficient to prove that $M \subset E$. Again we shall use the fact that $M \subset B$, for all $B \in A$ and show that $E \in A$.

Clearly $a$ is an extreme point of $M$, because there is no $x \in M$ with $x < a$, by Lemma A.5.

To show that $f(E) \subset E$, let $c$ be an extreme point of $M$. We need to show that $f(c)$ is an extreme point as well. Accordingly, let $x \in M$ with $x < f(c)$ (the task is to show $f(x) \leq f(c)$). But note that $M = M_c$, by Lemma A.6, so we get $x \in M_c$ and thus $x \leq c$ or $f(c) \leq x$, where the latter possibility is ruled out immediately ($x < f(c)$) so we get either $x < c$ or $x = c$. If $x < c$ then, by $c$ being an extreme point, we get $f(x) \leq c$ and, since $c \leq f(c)$, $f(x) \leq f(c)$ follows. If $x = c$ then, trivially, $f(x) \leq f(c)$. Thus $f(E) \subset E$.

It remains to show that $E$ satisfies the condition under (iii), in the definition of $A$ (then we’re done). Accordingly, let $T$ be a non-empty, totally ordered subset of $E$ and $b$ the least upper bound for $T$ in $A$. We need to prove that $b$ is an extreme point (of $M$).

Accordingly, let $x \in M$ with $x < b$. We want to show that $f(x) \leq b$. But first note that $b \in M$. Indeed, $T \subset E \subset M$. So $b \in M$, since $M \in A$ (by Lemma A.4.)

Second, we couldn’t have the case where $x$ is another upper bound for $T$ (this would render $b \leq x$, contradicting $x < b$). Still, since every $c \in T$ is an extreme point of $M$, it follows that $M_c = M$, for every $c \in T$, that is, for each $c \in T$, either $x \leq c$ or $f(c) \leq x$. Hence we must have at least one $c \in T$ with $x \leq c$, since otherwise $c \leq f(c) \leq x$ for all $c \in T$ which would render $x$ to be an upper bound for $T$, which, as pointed out, is not possible. So we do have some $c \in T$ with either $x < c$ or $x = c$. If $x < c$ then, by $c$ being an extreme point, we get $f(x) \leq c$, and thus $f(x) \leq b$. If $x = c$ then $x$ is an extreme point of $M$, so by Lemma A.6, $b \in M_x$. Hence $f(x) \leq b$ in this case also (otherwise $b \leq x$). Which concludes that $f(x) \leq b$, and the proof. □

We can now show that $M$ is a non-empty, totally ordered subset of $A$. Note that this concludes the proof of The Bourbaki fixed-point theorem (Lemma A.3) and as such this section.

A.8 Lemma. $M \in A$ and $M$ is totally ordered.
Proof. That $M \in A$ is already shown in Lemma A.4.

Let $x, y \in M$. By Lemma A.7, every element of $M$ is an extreme point of $M$. In particular, this is true of $x$, so by Lemma A.6, $M_x = M$ from which $y \in M_x$ follows. Hence we must have either $y \leq x$ or $f(x) \leq y$, where the latter renders $x \leq y$ since $x \leq f(x)$. In short: $y \leq x$ or $x \leq y$. This concludes the proof. □

A.2 From the Bourbaki fixed-point theorem to Zorns lemma

A.9 Lemma. Let $A$ be a non-empty, strictly inductively ordered set. Then $A$ has a maximal element.

Proof. Let $\leq$ denote the order of $A$. We need to show that there exists an $m \in A$ such that the set $A_m = \{ y \in S : m < y \}$ is empty. But suppose no such $m$ exist, then we get

$$A_x = \{ y \in A : x < y \} \neq \emptyset$$

for all $x \in A$, and thus – by Proposition 2.9 – that there exist some $f \in \prod_{x \in A} A_x$.

Clearly this $f$ is an increasing mapping in $A^A$, so the hypothesis of the Bourbaki fixed-point theorem is full-filled, which renders the existence of some $\xi \in A$ such that $f(\xi) = \xi$, contradicting the fact of $f(\xi) \in A_\xi$. Hence $A$ has a maximal element. □

A.10 Lemma. Let $(S, \leq)$ be an inductively ordered set. Then the set

$$\mathcal{A} = \{ T : T \text{ is a non-empty, totally ordered subset of } S \}$$

is strictly inductively ordered by $\subset$.

Proof. Let $\mathcal{F}$ be a non-empty, totally ordered subset of $(\mathcal{A}, \subset)$. We want to show that the set

$$Z = \bigcup \mathcal{F}$$

is a least upper bound for $\mathcal{F}$ in $\mathcal{A}$. But clearly, $Z$ is a least upper bound for $\mathcal{F}$ in $\mathcal{P}(S)$, so it remains to show that $Z \in \mathcal{A}$, i.e., that $Z$ is totally ordered by $\leq$.

Accordingly, let $x, y \in Z$. Note that there exists $X, Y \in \mathcal{F}$ with $x \in X$ and $y \in Y$. Moreover, since $\mathcal{F}$ is totally ordered by $\subset$, either $X \subset Y$ or $Y \subset X$. Either way, we get $x$ and $y$ to be member of one and the same set $F \in \mathcal{F}$. By $F$ being totally ordered by $\leq$, we must have either $x \leq y$ or $y \leq x$. Hence $Z$ is totally ordered by $\leq$ and a member of $\mathcal{A}$, which concludes the proof. □
Zorn’s lemma. Every non-empty, inductively ordered set \((S, \leq)\) has a maximal element.

Proof. Put

\[ A = \{ T : T \text{ is a non-empty, totally ordered subset of } S \}. \]

By Lemma A.10, \(A\) is strictly inductively ordered by \(\subset\). Hence by Lemma A.9, \(A\) has a maximal element \(M\). In particular, we have \(M\) totally ordered by \(\leq\) and \(\emptyset \neq M \subset S\). Hence \(M\) has an upper bound \(m\) in \(S\), since \((S, \leq)\) is inductively ordered.

It remains to show that \(m\) is a maximal element of \((S, \leq)\). But if \(x \in S\) with \(m \leq x\), then \(M \cup \{x\}\) is totally ordered by \(\leq\) and as such a member of \(A\). By \(M\) being a maximal element of \((A, \subset)\), we get \(M = M \cup \{x\}\) and, thus \(x \in M\) and \(x \leq m\). Hence \(x = m\), which concludes the proof. \(\square\)
Bibliography


Index

accumulation point, 20
almost all, 24
anti-symmetric, 16
argument, 10
axiom of choice, 15

base, 21
Bourbaki fixed-point theorem, 39
cartesian product, 9
choice function, 14
closed set, 18
closure, 20
cluster point, 31
codomain, 10
cofinal mapping, 28
collection, 11
compact, 22
composite mapping, 11
converge, 31
coordinate mappings, 14
directed set, 26
directed by reverse inclusion, 27
domain, 10
empty set, 7
eventually, 31
extension, 11
family, 11
family, indexed, 12
finite intersection property, 22
finite subcover, 22
frequently, 31
generalized cartesian product, 13
image, 11
increasing mapping, 39
indexed family, 12
inductively ordered, 17
interior, 20
inverse image, 11
least upper bound, 17
mapping, 10
maximal element, 17
member, 7
multiplicative axiom, 16
mutually disjoint, 9
neighborhood, 20
net, 27
net, based, 27
open cover, 22
open set, 18
ordered pair, 9
partial ordering, 16
power set, 9
product topology, 23
range, 10
reflexive, 16
relation, 9
restriction, 11
selection, 14
set, 7
  finite, 8
  infinite, 8
strictly inductively ordered, 17
subnet, 30
subproducts, 34
subset, 7
topological space, 18
topology, 18
  discrete, 18
  generated, 20
  trivial, 18
total ordering, 16
transitive, 16
upper bound, 17
value, 10
Zorn’s lemma, 17, 43