Knowledge and Writing in School Mathematics

Till min far Bengt och min kusin Annelie.
Jag vet att ni är med mig.

"What I write is different from what I say, what I say is different from what I think, what I think is different from what I ought to think and so it goes…"

Franz Kafka

Örebro Studies in Education 53

ANNA TELEDAHL

Knowledge and Writing in School Mathematics
A Communicational Approach

Cover picture: Eva Taflin

Title: Knowledge and Writing in School Mathematics
– A Communicational Approach
Publisher: Örebro University 2016
www.oru.se/publikationer-avhandlingar

Print: Örebro University, Repro 09/2016

ISSN 1404-9570
ISBN 978-91-7529-156-7

Abstract

Anna Teledahl (2016): Knowledge and Writing in School Mathematics – A Communicational Approach. Örebro Studies in Education 53, 129 pp.

This thesis is about young students' writing in school mathematics and the ways in which this writing is designed, interpreted and understood. Students' communication can act as a source from which teachers can make inferences regarding students' mathematical knowledge and understanding. In mathematics education previous research indicates that teachers assume that the process of interpreting and judging students' writing is unproblematic. The relationship between what students' write, and what they know or understand, is theoretical as well as empirical. In an era of increased focus on assessment and measurement in education it is necessary for teachers to know more about the relationship between communication and achievement. To add to this knowledge, the thesis has adopted a broad approach, and the thesis consists of four studies. The aim of these studies is to reach a deep understanding of writing in school mathematics. Such an understanding is dependent on examining different aspects of writing. The four studies together examine how the concept of communication is described in authoritative texts, how students' writing is viewed by teachers and how students make use of different communicational resources in their writing. The results of the four studies indicate that students' writing is more complex than is acknowledged by teachers and authoritative texts in mathematics education. Results point to a sophistication in students' approach to the merging of the two functions of writing, writing for oneself and writing for others. Results also suggest that students attend, to various extents, to questions regarding how, what and for whom they are writing in school mathematics. The relationship between writing and achievement is dependent on students' ability to have their writing reflect their knowledge and on teachers' thorough knowledge of the different features of writing and their awareness of its complexity. From a communicational perspective the ability to communicate [in writing] in mathematics can and should be distinguished from other mathematical abilities. By acknowledging that mathematical communication integrates mathematical language and natural language, teachers have an opportunity to turn writing in mathematics into an object of learning. This offers teachers the potential to add to their assessment literacy and offers students the potential to develop their communicational ability in order to write in a way that better reflects their mathematical knowledge.

Keywords: Mathematics, Writing, Students, Assessment, Communication

Anna Teledahl, School of Humanities, Education and Social Sciences. Örebro University, SE-701 82 Örebro, Sweden, ate@du.se

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Acknowledgements (in Swedish)

Ett förord till en avhandling skriver man bara en gång men förordsskrivande är ändå en egen skrivgenre med speciella regler. I vanliga fall skulle jag se det som självklart att strunta i dessa regler eller till och med se det som ett mål att bryta mot dem men nu är det annorlunda. I mina ögon skriver man ett förord för att man vill tacka personer som varit viktiga under skrivandet av avhandlingen och då är det viktigt att visa dem respekt. Mitt sätt att göra det är att hålla mig till de outtalade regler som finns om hur ett förord till en avhandling ska se ut. Förordet brukar inledas med att man liknar sitt avhandlingsprojekt vid en resa eller väljer någon annan metafor eller att man berättar en anekdot från sin tid som doktorand. Jag väljer metaforen. För mig är mitt avhandlingsarbete ingen resa utan ett trästycke. Jag började för fem år sedan med ett oformligt ämne i ek. Under åren har jag med allt vassare verktyg huggit, täljt och mejslat fram något som jag inte alls kunde föreställa mig från början. Till skillnad från många av mina doktorandkollegor hade jag en väldigt vag uppfattning om vad jag ville göra, säga eller undersöka. Nu när jag står här med slutresultatet är jag så glad över det jag har gjort, att det är färdigt och att det uttrycker något som jag tycker är viktigt.

Efter metaforen är det dags att tacka de som på olika sätt varit viktiga för mig. Det är kutym att inleda med att tacka sina handledare och det är förstås naturligt eftersom de, för mig liksom för de flesta doktorander, varit enormt viktiga på så många olika sätt. Man behöver stöd när man är forskarstudent och jag är tacksam över att jag verkligen har fått det från mina handledare, Johan Öhman och Eva Taflin. Tack Johan för att du alltid har sagt rätt saker, på rätt ställe och i rätt tid och för att du har visat ett sådant stort förtroende för mig hela tiden. Jag hoppas att vi får chansen att fortsätta arbeta tillsammans i framtiden. Tack Eva för att du har inspirerat mig i över 20 år, för att du trott på mig, puttat fram mig, diskuterat med mig och för att du alltid har mitt bästa för ögonen men allra mest av allt för att du varit och är min vän. Jag vill också tacka Ninni Wahlström som innan flytten till Linné var min handledare. Tack för att du satte ribban högt och litade på mig.

När man tackat sina handledare är det vanligt att man går vidare och tackar personer som varit inblandade i forskningen och i skrivandet. Jag inleder med att tacka alla lärare och elever som deltagit i mina studier men som jag inte kan nämna vid namn. Tusen tack för att ni ställt upp.

Jag har haft förmånen att vara med i en forskarskola under min doktorandtid. *Forskarskolan Teknikburna kunskapsprocesser*, som är ett samarbete mellan Högskolan Dalarna och Örebro universitet, har spelat en mycket stor roll i att jag uppskattat mina fem doktorandår så mycket som jag gjort. Jag har fått fantastiska vänner och upplevt så mycket roligt tillsammans med Eva, Mark, Anna, Megan, Sören, Olga, Giulia, Eva-Lena, Veronica, Edgar, Lena, Magnus, Isabella, Sana och Iryna. Jag vill tacka er alla för att ni gjort min doktorandtid till en av de bästa i mitt liv.

Inblandade i skrivandet och forskningen har också de varit som vid olika tillfällen och i olika stadier läst mina texter. Jag vill tacka Guy, Christian, Astrid och Per som varit officiella läsare vid seminarierna i Örebro för det ovärderliga stöd som er insiktsfulla och välvilliga läsning gett mig. Andra läsare som vid olika tillfällen läst mina artiklar har varit Maria B-H, Jorryt, Yvonne, Matilda, Ulrika, Eva-Lena, Lovisa, Sara och Maria O. Jag vill tacka er för att ni lagt tid och energi på att hjälpa mig. Kommentarer av kritiska vänner är oumbärliga i processen och utan er hade det inte gått.

Ordentligt inblandade i min forskning har också mina kollegor på matematikdidaktikinstitutionen på Högskolan Dalarna varit. Jag vill tacka er alla för att ni stöttat mig på olika sätt och för att ni fått mig att må bra. Tack Magnus J för att du haft ett sådant förtroende för mig. Tack Marit och Jeff för att ni stöttat mig och skärpt min kritiska förmåga. Tack Janne och Helen som delat med er av allt från er egen forskarskola. Tack Helena och Magnus F för kloka frågor och viktiga diskussioner. Tack Lovisa för goda råd och stöttning. Tack Johan, Kristofer, Maria S, Ingela, Andreas, Åke, David, Emma, Jonas, Lennart, Johanne, Hannele, Maria B-H, Lotta, Carina, Per-Eric, Sven, Fredrik, Torild och Kerstin för att ni varit kollegor som fått mig att känna att jag jobbar på världens bästa arbetsplats.

Det är svårt att skriva detta utan att det blir en uppräkning. Men eftersom det ingår i genren så fortsätter jag med att tacka andra som varit inblandade i min forskning. Jag vill tacka övriga kollegor på Högskolan Dalarna som varit viktiga för mig. Tack Stina för att du varit världens bästa chef. Tack Bengt för att du var den som från början såg till att jag blev doktorand. Tack Gunn för att du är så toppenbra att jobba med. Tack Gunilla för att du visade mig vägen. Tack (igen) Eva Hultin för att du spridit glädje och varit en inspiration i ditt ledarskap. Tack också till alla andra kollegor på Högskolan Dalarna som på olika sätt uppmuntrat mig och varit intresserade av det jag gjort.

De sista som har att göra med forskning som jag vill tacka är Maria S och Tomas på skolkontoret. Tack för att ni gjort min tid som kommundoktorand

till en härlig tid med stöd och uppmuntran. Tack också till alla doktorandkompisar som jag träffat under den här tiden. Tack Anette, Youkiko, Olof, Niclas, Kerstin, Jorryt, Yvonne, Linda, Miguel, Jonas, Richard, Andreas Ec, Andreas Eb, Helena R, Helena J, Malin, Andreas B och Maria L för allt roligt vi haft.

Efter att ha tackat alla som haft betydelse för forskningen är det dags att tacka alla som inte varit inblandade alls. Människor som varit svalt intresserade av min forskning men som istället sett till att jag har haft roligt när jag inte skrivit akademiska texter. Tack Sanna för att du finns, tack till mina två Malin för att ni sett till att jag blivit svettig och för att ni gett mig annat att tänka på, tack till Tobbe, Ullis, Jonas, Lena, Håkan, Anna och Johan för allt kul vi haft tillsammans. Tack till mina innebandyflickor och tränarkollegor för att ni gett mig annat att bry mig om än forskning. Tack alla mina gamla kollegor på Hgs med Claes i spetsen. Tack till min mor, mina syskon och deras familjer, mina svärföräldrar, Britt-Marie, min svåger och svägerska med familjer och mina fina grannar Carola och Göte. Ni struntar i om jag blir doktor i pedagogik och det känns skönt.

Det är relativt vanligt att man i förordet sparar de viktigaste människorna till sist. De som spelat allra störst roll för att det blivit någon avhandling över huvud taget. Det har också jag gjort. Jag vill rikta ett särskilt tack till några speciella människor. Egentligen vill jag inte tacka dem för något de gjort utan för att de finns i mitt liv men några saker nämner jag ändå. Tack Magnus Fahlström för att du varit min klippa och vän. Med aviga, långsökta eller pricksäkra metaforer, motion i mängder, oändligt tålamod, thailändsk mat, grekisk sprit och oändlig omsorg har du betalat den amerikanska kalkonen femtioelva gånger och den har somnat in för alltid. Tack Megan Case för att du är min vän, för att du öppnat så många nya världar för mig, för att du hjälper mig förstå mig själv och för att du gjort mig till en bättre människa. Att du dessutom på ett fantastiskt sätt språkgranskat mina texter och slösat med kommatecken är en liten sak i jämförelse med vad du betyder för mig. Tack Anna Annerberg för att du på riktigt stått bredvid mig hela tiden. Tack för att du varit och är den som vet mest av alla om min forskning. Tack för att du delat det jobbiga och det roliga och för att du gett mig en omistlig och självklar vänskap som är bland det mest värdefulla jag har.

Sist av allt vill jag tacka min familj Kenneth, Tova och Tim för att ni varit totalt ointresserade av mitt doktorerande. Vi fortsätter prata om annat hemma och ni fortsätter att vara viktigast i mitt liv <3

Kavala 14 augusti 2016

Articles

Article 1

Teledahl, Anna & Öhman, Johan (in review). The Logic of Communication in Competency Frameworks for Mathematics.

Article 2

Teledahl, Anna (2015). Different modes in teachers' discussions of students' mathematical texts. *Teaching and Teacher Education*, 51 (0) 68-76.

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Article 3

Teledahl, Anna (in review). How young students communicate their mathematical problem solving in writing.

Article 4

Teledahl, Anna (in manuscript). Digital and analogue writing in mathematics.

Introduction

This is a thesis about young students' writing in school mathematics and the ways in which this writing is designed, interpreted and understood. The starting point for my interest was a discussion during a seminar in the early years of my teaching career regarding an authentic classroom situation. The situation involved a drawing by a student in primary school. The drawing, which is displayed on the title page of this thesis, was drawn as a part of a mathematics activity. The drawing displays a number of weapons and some ammunition grouped together respectively in threes to creatively and accurately illustrate the equality 9+1=10. In the seminar we were told that upon presenting the drawing the student had been scolded by the teacher for 'playing' rather than completing the assignment. To me, the detailed and elaborate drawing, along with the description of the failed communication, sparked a career-long interest in communication in mathematics and in the different problems associated with it. During the course of this research project, meetings with teachers at different levels of the educational system have confirmed that many teachers share my interest. To them this issue is of critical concern and they take an interest in theoretical as well as practical questions regarding communication in school mathematics.

Language and communication are central to any mathematical activity or interaction in a mathematics classroom. Students communicate to learn and learn to communicate. Communication acts as a tool for the learning of mathematics but it also provides teachers with an opportunity to assess the learning of individuals. Communication is also a competence, skill or ability that can be conceptualized as separate from other competences. Without communicational skills a student may have difficulties articulating her ideas and learning from others. Writing is an important part of communication in mathematics, especially for assessment. Morgan (1998), who examined secondary students' investigative writing, claimed that students are unlikely to 'pick up' the linguistic knowledge and skills needed to master different kinds of writing in mathematics without explicit teaching. Such teaching, however, may also be unlikely, given that many teachers have never been explicitly taught how to write themselves (Morgan, 2001b). Teachers generally advise their pupils how to use mathematical vocabulary, notation, graphs, charts and diagrams accurately, but they have, Morgan claims, difficulties describing the ways in which com-

ponents such as these should be combined to construct, for example, a convincing proof, a concise definition, or an appropriate account of a problem-solving process. Part of the reason for this may be that writing is often viewed as one complex sign, which is assessed and judged, in its totality (Blommaert, 2013). Blommaert argues that this composite judgment can be disassembled to allow for the different components of writing being distinguished. Writing, Blommaert says, has infrastructural, graphic, linguistic, semantic, pragmatic, social and cultural components. In mathematics education, Burton and Morgan (2000) have shown that mathematical writing is often talked about as one single entity, in particular when research focuses on discrete features like algebraic notation. Language and its use have come to attract increased attention in mathematics education research (Morgan, Craig, Schuette, & Wagner, 2014; Sfard, 2014). This increased interest is associated with what can be described as a 'social turn' in research, a turn that represents a shift in perspective from the individual and her acquisition of knowledge to viewing thinking and meaning as products of social interaction (Lerman, 2000, 2006).

My particular interest is in the mathematical writing of young students, aged 9-12. Their writing is interesting to examine for a number of reasons. Firstly, it exhibits different characteristics than the writing of more mature students, in part because the formalization of mathematical language is only starting to develop during these years. Mavers (2010) argues that many discourses on the development of children's communicational skills have, despite much evidence to the contrary, focused on the transformation from simplicity to complexity and from incompetence to competence. The focus on what children cannot do tends to distract attention from the sophistication of what they actually know and can do, she says. This thesis adopts the perspective that young students are competent text-makers doing semiotic work and as such they attend to questions of *what, how, why* and *for whom* and make use of the resources available to realize their intentions. With this perspective young students' writing is taken seriously as a form of competent communication.

Secondly, young students' mathematical writing, along with its interpretation or assessment, has not been well researched. Traditionally, school mathematics has contained relatively little writing, which, for some students, has been part of its appeal (Morgan, 2001b). With the 'social turn', and other reform movements, that have encouraged greater use of com-

munication, both oral and written, different types of writing, including journal writing, investigative writing and the reporting of the results of problem solving have increased in classrooms around the world (Morgan, 2001b). Studies on students' written mathematical communication have tended to focus on students older than 12 (see for example, Albert, 2000; T. S. Craig, 2011; Liu & O'Halloran, 2009; O'Halloran, 2005; Shield & Galbraith, 1998). The studies that focus on younger students' writing are often investigating the mathematical ideas expressed in writing rather than the writing itself (see for example Saundry & Nicol, 2006; Smith, 2003). This thesis focuses on the writings of young students, as products, mathematical texts, that are doing communicative work, and as such can be seen to reflect the communicative competence of their creators.

Another reason for investigating students' writing is that assessment in school mathematics has always relied heavily on students' written work (Morgan, 2001b). Mathematical texts produced by students act as important sources of information on students' achievements for both summative and formative purposes. The way these texts are designed and read will therefore have consequences for the teaching and learning as well as the assessment of mathematics. Morgan has shown that (1998) that teachers, the most frequent audiences for students' mathematical texts, tended to view these texts as transparent records of students' intentions as well as their understandings and cognitive processes. She also identified a general assumption that the act of writing and the process of interpreting and judging students' writing are unproblematic. Such a simplistic perspective on communication is questioned within research traditions such as social semiotics and discourse theory. In these traditions, language is viewed as constructive and situated, originating in social action, and, therefore, open to a number of interpretations (Gee, 2011; Kress, 2011; Wetherell, 2010b).

A final reason for the importance of investigating students' writing is that there are different types of writing in school mathematics. When solving mathematical problems, students are using writing for cognitive purposes as well as for presenting their work (Stylianou, 2011). Writing offers an opportunity to organize, visualize, systematize and manipulate different parts of the problem. Young students create ad-hoc representations to help them solve a particular problem where their representations are not intended for a general use, hence it is difficult to interpret their records as

evidence of general abilities (Smith, 2003). Students' documentation of their problem solving is often used by teachers as a convenient and lasting record of their achievements, which means that even in situations where they have used writing to communicate with themselves, students are still also communicating with their teachers (Morgan, 2001b). The two processes of writing, for oneself and for others, have tended to merge in school mathematics (Morgan, 1998). This is a particularly salient feature in the documentation or reporting of problem solving. In problem solving, students are expected not only to produce a correct answer but also to provide a record of their problem-solving process; hence, such texts are richer and more varied than many other texts in mathematics. This is why texts from young students' mathematical problem solving were chosen to form the empirical material for this thesis.

In summary, it can be argued that, given that assessment in mathematics tends to rely on students' writing, the relationship between written communication and achievement is important, and there is reason to believe that this relationship is unproblematized and taken for granted. Evidence for this can be found in policy documents, in practice and in research. The central problem of this thesis is, therefore, that the highly complex relationship between what a student writes and what she knows or understands tends to be regarded as unproblematic in mathematics education.

Thesis aims

The purpose of the thesis is to examine and problematize students' writing in school mathematics and the various understandings of the relationship between students' written communication and their achievement. This relationship is a theoretical as well as an empirical problem. There are different ways in which the nature of communication can be understood, and, consequently, there are also different ways to relate to communication in educational practice. In an era of increased focus on assessment and measurement in education, I argue that it is necessary to know more about the relationship between communication and achievement, theoretically as well as empirically. To add to this knowledge, the thesis has adopted a broad approach, and the thesis project consists of four studies, each of which has been presented in the form of an article. The aim of these studies is to reach a deep understanding of writing in school mathematics. This understanding is dependent on examining different aspects of writing. The four studies together examine how the core concept of com-

munication is described in authoritative texts, how students' writing is viewed by teachers and how students make use of different communicational resources in their writing. The studies correspond to the following research questions:

A. What communicational logic is embedded in international authoritative texts in mathematics education, and what are the possible consequences for teaching and learning?
B. How do teachers interpret, understand and assess students' mathematical writing?
C. How do students use different communicational resources in their mathematical writing?

Together these are thought to cover aspects of writing that are important for problematizing the relationship between written communication and achievement in mathematics. Three of the four studies are set in a Swedish context, but the first study examines competency frameworks which have an international reach. The results of the studies, as well as the thesis, are hoped to be relevant for mathematics educators, anywhere, who have to make subjective assessments of student work. The four studies and their interrelations are described next.

The four studies

Students' writing in school mathematics is embedded in a cultural and social context which includes ideas about the purpose of writing and what constitutes good writing, as well as theories on the nature of communication, both in general and in mathematics in particular. Authoritative texts such as steering documents and frameworks are part of this cultural and social context, and they comprise the ideology and logic of the practice in which they are written (Östman, 1995). Given that communication is not an unequivocal concept and that different conceptions or logic about the nature of communication affect the teaching, learning and assessment of mathematics, it is important to examine authoritative texts that influence mathematics education. The first study, therefore, investigates competence frameworks for mathematics with the aim of identifying their communicational logic. The way written communication is conceptualized affects the way teachers use students' writing in their teaching and assessment of mathematics. The second study is, therefore, focused on teachers.

As the most common readers of students' texts, teachers interpret, understand and assess these texts as part of their teaching and assessment practices. Such practices are complex, and the meanings that teachers construct from a text will depend on their individual resources and previous experience as well as on the ways in which the teacher collective, rather than the individual teachers, interprets ideas and concepts. Such collective ideas and patterns of interpretation of students' writing contribute to forming the cultural context in which this writing takes place. The aim of the second study is, therefore, to examine mathematics teachers' ways of collectively interpreting and discussing students' mathematical texts. The way teachers talk about students' texts, and the features and aspects they focus on, is part of their assessment literacy. Microstudies that focus on how teachers assess particular tasks or assignments are not common in assessment research. Teachers' assessments of texts can focus on content, such as mathematical strategies, but they can also focus on communicational aspects such as coherence, logic and comprehensiveness. In order for students' writing to develop, its various features must be identified and highlighted. The third and the fourth studies, therefore, aim to examine authentic mathematical texts from young students.

In their writing in school mathematics, students draw on a variety of different communicational resources in their design and creation of mathematical texts. Formal mathematical language is complemented with students' natural language as a way to explain, organize and give structure to the formal writing. The integration of these two different languages in connection with mathematical problem solving plays an important role in how this writing can be interpreted and understood. The aim of the third and fourth studies is to disassemble students' writing by examining the communicational resources that students draw on and the way these resources are integrated in the design of mathematical texts in connection with problem solving.

The focus of the thesis moves from the broader cultural context, in which authoritative texts with various logics concerning communication affect students and teachers, to the way teachers interpret, understand and assess students' mathematical writing, to the immediate context, in which students create and design their texts with the communicational resources available to them. Below the four studies are described in brief.

Study 1: The Logic of Communication in Competency Frameworks for Mathematics

The first study corresponds to the first research question: *what communicational logic is embedded in international authoritative texts in mathematics education, and what are the possible consequences for teaching and learning?* The study aims to explore and problematize the concept of communication in mathematics education. In doing so the study serves as a theoretical anchor to the thesis while at the same time offering an opportunity to identify critical issues concerning communication in mathematics education. The concept of communication is explored and problematized through the investigation of three internationally renowned competency frameworks in mathematics. Competency frameworks in mathematics are constructs which categorize the cognitive skills and abilities that students use when they learn or do mathematics (Kilpatrick, 2014).

The frameworks examined are The PISA 2012 Mathematics Framework (OECD, 2013), The Singapore Mathematics Framework (MES, 2012) and The Common Core Standards for Mathematical Practice (NGACBP, 2010). These frameworks are chosen because: a) they explicitly address communicational ability as an ability which is separate from other abilities; b) they are either listed, or build on other frameworks that are listed, as influential competency frameworks in The Encyclopedia of Mathematics Education (Kilpatrick, 2014); c) they speak to different audiences; d) they are all available online; and e) they are all in English. It can thus be argued that these frameworks have a global reach and that they influence mathematics teaching and assessment at different levels in mathematics education in a number of countries.

The study adopts a discourse-analytic approach to text analysis, and the concept of logic is used to capture the presuppositions about communication that underpin the frameworks. Logic, a concept developed by Glynos and Howarth (2007), refers to the rules and grammar of a particular practice. In the analysis, analytical questions regarding with whom, with what and how students communicate, are used to identify these rules.

The result of the first study indicates that the three frameworks operate with a logic that casts mathematical communication as being both transparent and unproblematic.

The first study is reported in an article submitted to the *Journal of Curriculum Studies*.

Study 2: Different modes in teachers' discussions of students' mathematical texts

The second study corresponds to the second research question: *how do teachers interpret, understand and assess students' mathematical writing?* The aim of the study is to examine the ways in which mathematics teachers discuss students' mathematical texts. The study takes a discourse analytic approach and the object of study is teachers' collective discussions rather than their individual conceptions. The idea that a particular professional collective may share some of the different ways in which they interpret different phenomena is a cornerstone of discourse theory (Wetherell, 2010a).

The study set out to identify the approaches to interpreting, understanding and assessing mathematical texts that were visible in the discussions. Group interviews were conducted with 19 middle school teachers who were presented with, and asked to discuss, 15 different mathematical texts produced by students in grade four. The transcriptions from the interviews were analyzed through a combination of quantitative summative content analytic and discourse analytic approaches.

The results indicate that two different modes are visible when teachers discuss the mathematical texts. The first is a pedagogical mode connected to the teachers' roles as teachers or pedagogues and the second is an assessment mode which is connected to teachers' roles as examiners.

The second study is reported in an article published in the journal *Teaching and Teacher Education*.

Study 3: How young students communicate their mathematical problem solving in writing

The third study corresponds to the third research question: *how do students use different communicational resources in their mathematical writ-*

ing? The aim of the study is to examine young students' mathematical writing. By disassembling this writing it is possible to identify students' choices and employment of communicational resources to document and communicate their problem solving.

A sample of 519 texts from students aged 9-12 was collected from ten teachers, from eight different Swedish schools, whom all had agreed to collect and forward accounts of problem solving, i.e. mathematical texts, from their student groups. The problem type was a form of linear Diophantine equation that involved distribution of, for example, legs on animals, wheels on vehicles or computers in boxes. The method of analysis of the study combines elements from multimodal discourse analysis (Jewitt, 2011a) and conventional qualitative content analysis (Hsieh & Shannon, 2005). A multimodal analysis takes young students' communication seriously by accounting not only for the different modes but also by analyzing to what uses these modes are put.

The findings of the study indicate that students have access to and make use of a number of communicational resources as they attend to questions such as how, what, for whom and why they are writing. The great diversity indicates that even students from the same group have very different ideas about these questions.

The third study is reported in an article submitted to *International Journal of Mathematical Education in Science and Technology*.

Study 4: Digital and analogue writing in mathematics

Like the third study, the fourth study also corresponds to the third research question: *how do students use different communicational resources in their mathematical writing?* This study aims to compare students' use of various communicational resources in their design of analogue and digital texts. The study examines seven 12year-old students' documentation of problem-solving processes, of which two were recorded digitally using an interactive white board (IWB). The data collection was organized in collaboration with a teacher who had participated in an earlier research project, and resulted in 28 mathematical texts, of which 14 were digital. The students were also interviewed in connection to their digital problem solving.

The mathematical texts were analyzed through multimodal discourse analysis with the aim of identifying the different communicational resources used by the students. The analysis also included searching for differences and similarities between the ways resources are employed in the design of analogue and digital texts.

The findings of the fourth study indicate that there are few, but potentially important differences, between the digital and the analogue texts. When viewed as products for communication the digital texts contain less elements such as transition markers, explanations and structuring devices that serve to facilitate the reading of the text and as a result of this the texts also display less internal coherence than their analogue counterparts.

The fourth study is reported in a manuscript yet to be submitted.

Previous research

This section is meant to give a background to the field of mathematical writing as well as to introduce previous research on students' writing in school mathematics. Given that students' writing is used as a base from which inferences about students' achievement can be made, the section includes a presentation of research on teachers' assessment competence in general and in relation to the assessment of students' mathematical texts in particular. The section is opened with a description of the procedure with which the previous literature has been found.

Literature search

As with most long-term projects, the literature that provide the background for this thesis as well as the background for the four articles was discovered and included at several different stages over the course of five years. Some phases of the literature search have been systematic, while others have not. A detailed description of when and how every article, book or book chapter was found is impossible, but the major search routines, key terms for searching and criteria for inclusion are described below.

Search strategies

Three main strategies were adopted to search for relevant literature: the use of online databases, the snowball method, and searching current and archived issues of particular journals. The databases and search engines used include Summon, EBSCO, Scopus and Web of Science. Searches have also been done using Google and Google Scholar. The so-called snowball method, in which the reference lists of key articles or books are used to identify relevant literature, was used during the entire project. With this method it was possible to identify a particular body of research in which scholars partly refer to each other's work. Two of the most prominent journals in the field of mathematics education, *Journal for Research in Mathematics Education* and *Educational Studies in Mathematics*, were selected for an online manual search in which each issue, dating from January 2000 to October 2014, was screened.

Key search terms

The search terms for various searches include *multimodality*, *writing*, *representation(s)* and *communication,* in combination with *student(s)*, *pupil(s)*, *children(s)*, *teacher(s)*, *assessment*, and *mathematics* or *mathematics*

education. The search for literature on *teachers' assessment literacy* was a separate search using the compound term in combination with *assessment competence*.

Criteria

The criteria for inclusion are divided into two parts. The first deals with students' writing and the second with teachers' assessment literacy. The search has included studies as well as literature reviews and theoretical articles that have specifically discussed the topic of students' writing or assessment literacy. Theoretical articles were included to provide a general background to the field.

The focus of the first part of the search was *young students' writing in mathematics*. The term *young* was defined as including ages 6 to 13, but studies involving older students were included if they dealt specifically with students' writing in a way that was deemed relevant. The term *students* was complemented with the terms *pupils* and *children* and together they were defined as young people in a school situation while *writing* was defined as all the documentation, recording, visualization and communication that students do using pen-and-paper or digital devices. Writing was not limited to any particular form of representation or to a particular function. *Mathematics* was defined as being a school subject, either as context or content.

In the second part the term *teachers' assessment literacy* was treated as a unit and it was complemented with the term *assessment competence* and from this search the snowball method was used. The specific connection to mathematics was included from a previously known article.

Students' writing in mathematics

Morgan (2001a) has argued that the written mathematical work of students in school mathematics typically serves two very different functions. It can be seen as a part of a learning process in which writing is used to record and perhaps reflect on various mathematical ideas; hence, the text is written by and for the student herself. It can also however, be seen as a product for the purpose of assessment; hence, written for a teacher or examiner. Unlike the work of professional mathematicians, which is often thought to be the model for school mathematics, the work in school mathematics often serves these two functions at the same time (Morgan,

2001a). Previous research on these two different functions is described below.

Writing to learn – writing for oneself

There are different purposes for students' writing in school mathematics. Students write mathematical texts of different lengths in response to different tasks and addressed to different readers. In connection with mathematical problem solving, a typical recipient for student writing would be the student herself. When investigating middle school students' use of representations in mathematical problem solving, Stylianou (2011) found that they create representations for themselves and for others. When problem solving is viewed as an individual cognitive activity, students use representations as tools towards the understanding, exploration, recording, and monitoring of their own problem solving. In the social context of the classroom, students use representations for the presentation of their work as well as to negotiate and co-construct shared understandings with peers. That students' writing or design of representations in school has a cognitive function in which students use their writing to record, visualize and organize for example their problem-solving processes, has been recognized by for example Izsak (2003), Goldin & Shteingold (2001) and Duval (2006).

Morgan (2001a) has noted that the cognitive function and the social function, in which students share their writing with others, tend to fuse in school. Where professional mathematicians' work is the result of what in school could be referred to as a write-up, the work in school mathematics is seldom written up; instead the problem-solving process and the writing process are integrated and the writing is expected to serve a personal function and a communicative function at the same time. This integration of public and private was also reported in a study by Fried and Amit (2003) in which they investigated the use of notebooks in two mathematics classrooms. Their study of the public and/or private character of notebooks concludes that a mathematics notebook, although partly belonging to the private domain, was treated as a public object and, as such, it may, at any time, serve as a text to be assessed.

Another type of writing that can be seen as personal for students is journal writing or expository writing (Shield & Galbraith, 1998). Such writing is thought to encourage students to reflect on their learning, and several

studies show that this kind of writing can be beneficial to students' mathematical understanding, problem-solving skills and attitudes (Bell & Bell, 1985; Bicer, Capraro, & Capraro, 2013; Borasi & Rose, 1989; T. S. Craig, 2011; Pugalee, 2001, 2004; Reilly, 2007). Kenyon (1989) went one step further by arguing that writing not only enhances problem solving, but rather that it *is* problem solving. He claimed that writing practice employs cognitive processes that are equal to successful problem solving, making it an ideal tool for problem solving. Kenyon defines writing as involving, planning, composition and revising. Mendez and Taube (1997) compared this process to the well-known steps of problem solving proposed by Polya (2008)—understand the problem/devise a plan/carry out the plan/look back—seeing obvious similarities.

Studies that have investigated students' writing have adopted a variety of methodological approaches and theoretical frameworks. Several different schemes for analyzing students' writing have been proposed. Clarke, Waywood and Stephens (1993), through a study of 500 Australian students, aged 11 to 17, developed a scheme for describing students' journal writing consisting of three categories; 'recount', 'summary' and 'dialogue'. Craig (2011) later successfully applied this scheme to the writing of students in a university mathematics course. In a study involving students aged 12-13, Shield and Galbraith (1998) developed a coding scheme for analyzing the students' expository writing that focused on the components of explanation through an exemplar – a worked example of a procedure.

Writing to provide opportunity for assessment – writing for others

Assessment in mathematics has been the object of study in a large body of research that has dealt with the *how, what, when, who, where* and *why* of assessment. Much of the mainstream thinking on assessment rests on the principle that students possess certain attributes such as skill, knowledge, ability and understanding and that the main purpose of assessment is to discover, and if possible, measure these (Morgan, 1999). When the purpose of assessment is the discovery of such attributes, it becomes concerned with concepts like validity, reliability and objectivity, which are all concerned with coming as close as possible to the 'truth', i.e. a true and accurate understanding of the attributes of a particular individual or a group of individuals. Morgan argues that this positivist tradition is particularly strong in mathematics, given the discipline's focus on right or wrong answers. In order to discover the attributes, however, those doing

the assessment must rely on the students' verbal and/or written communication to inform their judgment. Morgan (1999) argues that although mathematics educators have widely accepted constructivist ideas in relation to how students make sense of mathematical activity, there is still a naïve understanding of communication as mere 'transmission' when it comes to assessment. The transmission metaphor implies that meaning resides within the text where it accurately reflects the intentions of the author, and it is, thus, the work of an examiner to extract this meaning. In her critique of this view, Morgan draws on contemporary theories of communication, such as social semiotics and discourse theory, when she claims that there are multiple ways in which a text can be read and that there is no simple correspondence between these readings and the intentions of the author of the text.

Research on students' writing in school mathematics, widely used in Australia, Oceania and South East Asia, include what is referred to as Newman research. Newman had proposed a process for students' work on pencil-and-paper text items that included: reading the question, comprehending what is read, carrying out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy, applying the process skills demanded by the selected strategy and encoding the answer in an acceptable written form (Ellerton & Clarkson, 1996). Errors in students' answers were thought of as having arisen from problems in one or several of these separate processes. Later Newman added a composite category that he termed 'careless' to account for unknown factors. Newman research, Ellerton and Clarkson claim, has generated evidence from numerous studies that suggest that it is far more common for children to experience problems with semantic structure, vocabulary and mathematical symbolism than they do with, for example, standard algorithms (Ellerton & Clarkson, 1996). Much of the research that focuses on students' written solutions to word problems has tended to focused on the mathematical mistakes that students make (see for example Knifong & Holtan, 1976), whereas others have focused on the students' reading skills (see for example Bergqvist & Österholm, 2012; Österholm, 2006).

Studies on young children's writing in connection to problem-solving activities have focused on their use of representation. Saundry and Nichols (2006) suggested that children may use, for example, drawing for different

purposes, "drawing as problem-solving" as well as "drawing of problem-solving". Smith, who also investigated children's use of representations, claimed that the representations that students use act as resources to solve particular problems (Smith, 2003). Children who are allowed the freedom to create their own idiosyncratic representations are likely to create them ad hoc to solve particular problems. They are, thus, not attending to the goal of solving any problem but rather a particular one (Smith, 2003). For this reason, looking at these texts as representations of students' understanding, ability to generalize, or ability to deal with abstractions, may be misleading.

As with the studies mentioned above, studies that have investigated students' writing for others have also adopted different methodological approaches and theoretical frameworks. Morgan (1998) used a discourse analytic approach in her analysis of secondary students mathematical texts through the meta-functions, suggested by Halliday (1978), the ideational, the interpersonal and the textual (see the theory section, p. 39 for a more comprehensive description of these meta-functions).

Teachers' assessment literacy

Assessment literacy refers to an understanding of fundamental assessment concepts and practices that are likely to influence educational decisions. In recent years a number of professional development programs for teachers have focused on assessment literacy (Popham, 2009, 2011). Given internationally increased focus on assessment and measurement paired with an increased, externally imposed, scrutiny of schools, it is easy to understand why assessment literacy might be regarded as an advisable and relevant target for teachers' professional development (Biesta, 2010; Popham, 2009). In 2001 Brookhart reviewed research on teachers' assessment competence and skills connected to "Standards for Teacher Competence in the Educational Assessment of Students" (an American effort to establish standards for teachers' knowledge about educational assessment). The studies examined had investigated teachers' knowledge through surveys of teacher attitudes, beliefs, and practices, tests of assessment knowledge and reviews of teachers' assessments themselves. The review concluded that most studies suggest that teachers need more instruction in assessment (Brookhart, 2001).

Popham (2009) argues that many teachers today know little about educational assessment. He also argues, however, that considering how infrequent the concepts and practices of educational assessment have been featured in teacher education, the gap in teachers' assessment-related knowledge is understandable. Research that has investigated teachers' assessment practices has also criticized such practices for failing to meet standards of reliability, objectivity and validity (Allal, 2012). Research also suggests that teachers themselves feel inadequately prepared to assess their students' performances (Mertler, 2004).

Assessment literacy has been described as involving a number of different practices. The Standards for Teacher Competence in the Educational Assessment of Students mentioned above emphasized

> choosing and developing assessment methods appropriate for instructional decisions; administering, scoring, and interpreting the results of externally produced and teacher produced assessment methods; using assessment results when making educational decisions; developing valid student grading procedures which use assessments; communicating assessment results to students, parents, and other lay audiences and educators; and recognizing unethical, illegal, and otherwise inappropriate assessment methods and uses of information. (Brookhart, 2001)

McMillan (2000) has also summarized fundamental assessment principles for teachers and school administrators:

> Assessment: is inherently a process of professional judgment; is based on separate but related principles of measurement evidence and evaluation; is influenced by a series of tensions; influences student motivation and learning; contains error. Good assessment: enhances instruction; is valid; is fair and ethical; uses multiple methods; is efficient and feasible; appropriately incorporates technology (McMillan, 2000).

Particularly important for this thesis is the element of interpretation in teachers' assessment of students. Teachers interpret observed test results or other types of information to come to a conclusion about a student's level of knowledge or skill. Such a conclusion may be referred to as *inference*, and although some inferences can be made with more confidence than others, no conclusion about a particular student's knowledge or skill can ever be made with certainty (Cizek, 2009). Even the most carefully collected information can lead to inferences that are invalid.

This thesis is about texts that young students produce in school mathematics. These texts are used to infer the extent of the students' mathematical knowledge and skills. Morgan and Watson (2002) argue that when assessing such texts produced by students, teachers rely on their professional judgement, which may be seen as part of the assessment literacy described above. As teachers read and assess students' texts, their professional judgment is formed by a set of resources which varies with teachers' personal, social and cultural history as well as their relation to the particular discourse. These resources are individual, as well as collective, and they include:

1. Teachers' personal knowledge of mathematics and the curriculum, including affective aspects of their personal mathematics history.
2. Teachers' beliefs about the nature of mathematics and how these relate to assessment.
3. Teachers' expectations about how mathematical knowledge can be communicated. Individual teachers may also have particular preferences for particular modes of communication as indicators of understanding. Thus, what appears salient to one teacher may not to another.
4. Teachers' experience and expectations of students and classrooms in general.
5. Teachers' experience, impressions, and expectations of individual students. (Morgan & Watson, 2002)

The third point captures the central question in this thesis. As teachers assess students' writing in school they draw on a number of resources of which their conceptions of communication is one. Morgan (1998) showed how the teachers she interviewed interpreted the meaning of the same passages of texts, produced by secondary students in mathematics, very differently. From the interviews she also concluded that teachers not only tended to view students' mathematical texts as transparent records of students' intentions as well as their understandings and cognitive processes, but also that the act of writing and the process of interpreting and assessing students' writing are phenomena that teachers think should be taken at face value. In relation to the discussion on teachers' assessment literacy above, such conceptions are connected to the question of validity. When it is not possible to know for certain, teachers have to strive to vali-

date their inferences by trying to "ascertain the degree to which multiple lines of evidence are consonant with the inference, while establishing that alternative inferences are less well supported" (Messick, 1988, p. 13). If teachers would question a straightforward relationship between students' writing and knowledge this would perhaps increase support for alternative inferences, thus making judgements less valid.

Summary

Previous research that has focused on the writing that students do in school mathematics for the purpose of learning have suggested that such writing is indeed beneficial to students. Several studies also indicate, however, that this writing, albeit created for personal use, may also be used as an object for assessment. These two purposes of writing are parallel to the conflict between assessment for formative purposes and assessment for grading purposes. Earlier research on students' writing for the purpose of assessment has been critical of the idea that such writing would accurately reflect the knowledge and intentions of its author. This critique constitutes an important motive for this thesis, for if the writing of students is taken as evidence of their general mathematical knowledge, it is important for teachers, as well as students, to have a thorough knowledge of this writing, about its different aspects, and about different ways of interpreting it. This is also strengthened by research that suggests that rather than acting as a simple transfer tool, writing poses a number of problems for students. Evidence suggests that students are more likely to experience problems with issues such as semantic structure, vocabulary and mathematical symbolism than they are with the mathematics itself. Young students have different approaches to mathematical writing, and, taken together, the research presented above suggests that this writing is in need of further research.

Research on teachers' general assessment literacy suggests that teachers need to know more about educational assessment. Studies also indicate that teachers are dissatisfied with their assessment practices and feel inadequately prepared to assess their students' performances. Teachers' assessment practices have also been criticized for failing to meet standards of reliability, objectivity and validity. In mathematics education research has shown that teachers can reach very different interpretations of students' writing and that they assume that interpreting this writing is unproblematic. Teachers are not questioning a straightforward relationship between

students' writing and knowledge, but rather assume that students' mathematical texts are transparent records of their intentions as well as their understandings and cognitive processes.

Theory

Theory, although thought to be very important to all research, is not a single uncontested concept in educational research (diSessa & Cobb, 2004; Niss, 2006). Theory can be used as a tool in, or serve as the object of, research (Sriraman & English, 2010). When used as a tool, theory can serve different purposes in research: it can provide a lens, or a set of lenses, through which a phenomena can be viewed and approached, it can be used for organizing a set of specific observations and interpretations, it can provide appropriate terminology and it can offer a research methodology (Niss, 2006). When used as a tool to describe, interpret, explain and justify observations, theory also has a strong influence on what is observed and what is omitted (Presmeg, 2010). Mathematics education is a field that has borrowed theories from several other academic disciplines and, given the complexity of teaching and learning, there is no single imported theory that encompasses all aspects of mathematics education (Lerman, 2010; Niss, 2006). Like mathematics education, the broad approach of this thesis in which the object of study includes the communicational logic of competency frameworks in mathematics, teachers' collective discussions of the assessment of students' mathematical texts and examples of students' writing, requires several theories that serve different purposes. In this section theories are presented along with definitions of different concepts, and the section as a whole serves to provide the perspectives, background, terminology and definitions necessary to understand the different aspects of students' writing that are central to the thesis. Theories of communication, which are presented first with a brief description of their historical development, provide a set of lenses through which the phenomenon of communication can be viewed. The theory of social semiotics is presented and serves to situate the last three studies in a perspective that views communication as inherently social, as well as to provide the terminology for talking about students' writing. Writing as a phenomenon is described in this section as a way of highlighting the aspects of writing that are important for understanding its relation to learning and assessment. That students can exhibit different kinds of communicational competence is a fundamental assumption in the thesis, which is why the theory behind the concept of communicational competence is briefly presented. This theory offers a perspective that serves to explain the basis on which students' communicational choices are made. The concepts of mathematical communication, mathematical language and mathematical writing can

be understood in different ways. These ways are described and serve to highlight the central relationship between mathematics and communication. The section concludes with a presentation of how the concept of mathematical literacy has been defined in research. This section serves to give insight into the complexity of defining what it means to know mathematics.

The historical development of different concepts of communication

The word *communication* is derived from the Latin word *communicare*, which means 'to share' or 'to make common' (Cherry, 1978). Communication studies have roots in ancient Greece from which they developed within several different research disciplines. Early theories of communication presented a rather simplistic model of the process of communicating (Mangion, 2011). In modern times this model is often referred to as the sender-receiver or transmission model and it constitutes the most elementary of the models of communication. A more complex version of the sender-receiver model introduces the concept of coding. The sender encodes a message which the receiver has to decode. Decoding involves interpretations regarding what the sender intended to say, which implies some sort of shared understanding. This way of thinking about communication as a process that is dependent on the sender's ability to code a message and the receiver's ability to decode it often presents language as a system of referents which mirrors the world. Here there is a correspondence between the world as it is and the language, but it is important to recognize that language is separate from the world. Words are used to label objects, material as well as abstract, and in this way the meaning of a word is the object that it names. This idea was influenced by the Biblical narrative in Genesis in which Adam names the objects of the world, and it was deeply rooted in the culture of the 18th and early 19th century (Eco, 1987; Mangion, 2011). Research in linguistics during this time was also influenced by this nomenclaturist theory, and comparative studies in which different languages were compared with the aim of finding the origin, and thus the true meaning, of words were common. Thinkers in philosophy such as Frege and early Wittgenstein proposed models for an objective understanding of language where meaning is instead defined in terms of objective truth conditions (Habermas & Cooke, 2000). The scientific approach to studying language, adopted by, for example, Wittgenstein, was also applied by the Swiss linguist Saussure, who aimed to estab-

lish linguistics as a scientific discipline that was more than just speculative (Mangion, 2011). Saussure was influenced by the comparative studies of different languages, but it was the differences rather than the similarities between languages that led him to reject the claim that language mirrored the world. Instead, he argued that language is a system of signs that in turn generate meaning. In Saussure's terminology a sign is a combination of the sound and the graphic inscription of a word (the signifier) and its meaning (the signified). Meaning, Saussure argued, is not inherent in the words but rather in their relations to other words. At around the same time that Saussure presented his model in a European context a closely related theory of signs was presented by Peirce in an American setting. A number of scholars in fields such as social semiotics, sociolinguistics, discourse theory and pragmatism (see for example Cherry, 1978; Dewey, 1929; Eco, 1987; Gee, 2008; Gumperz, 1982, 2010; Hymes, 1972; Kress, 2010a; Van Leeuwen, 2005) have, at various stages, criticized static models of language and communication such as the early ones described above for failing to account for the social influence on human communication. Scholars in these fields all became concerned, in different ways, with the social aspect of communication and, hence, it is possible to talk about a "social turn" in research on language and communication. This view places linguistic agency in the hands of individuals and recognizes the consequences of different choices in linguistic interaction (Austin, 1962; Kress, 2010a). That communication requires work, and that work changes things, suggested that speakers can *do* something with language, and that communication is not only a mechanistic reproduction of something already made common but rather the joint production of something new (Dewey, 1929). The irregularities, i.e. different pronunciations, different use of words, different structuring of sentences, which in other theories sometimes had been treated as mere "noise" in an otherwise ideal use of language, were instead believed to play an important role in human interaction. Research in a number of areas showed how speakers in different circumstances used language in a variety of ways to navigate in and manage their social environment, thus demonstrating a competence that went well beyond syntax or grammar (Austin, 1962; Gumperz, 1982, 2010; Kress, 2010a). A key thinker behind the idea of language as dynamic and social was Wittgenstein, who in his later works used the metaphor *language game* to refer to the way the use of language is governed by rules that are formed in interaction between people and where words and utterances gain their meanings from the social context (Öhman, 2006).

That language *does* something was also recognized as a source of power, and disciplines such as critical linguistics and critical discourse theory developed methods for understanding how power can be maintained and distributed in language and communication (Fairclough, 2003; Van Dijk, 2010). The different disciplines that took interest in the social in explaining communication all represent a significant move away from the simple sender-receiver model for communication. Pragmatics, sociolinguistics and social semiotics are examples of disciplines which have focused on how the formal properties of language can be used to explain different interpretations of speech and text. Like Wittgenstein, scholars in these disciplines assert that meaning is derived by context (Halliday, 1978). Although they agree that meaning resides not in the language itself but in its relation to the social context in which it is used, they differ in how they account for the social and in the kind of questions they ask.

Although research and theory have come a long way from the sender-receiver model, some scholars argue that this model still has the most fundamental effect on our way of thinking about communication (Reddy, 1979). Its core idea, Reddy claims, is still influencing the way people in general think about communication. This influence comes mainly from the way communication is talked about in language. Reddy uses the conduit metaphor to refer to the idea that the process of communication involves transferring human thoughts and feelings and that this entails a sender and a receiver. The conduit metaphor is echoed in the language itself in the form of metaphorical expressions such as "Try to *get* your *thoughts across* better" and "You still haven't *given me* any *idea* of what you mean" (Reddy, 1979, p. 286; italics original). In these and other metaphorical expressions, communication is still conceptualized as a process that involves different ways of packing, shipping and unpacking these thoughts. Reddy goes on to show that language is filled with expressions that allude to the process of successfully or unsuccessfully filling or packing (loading, inserting, capturing) some sort of container (words and sentences) with a message (ideas, thoughts and feelings).

Contemporary communication theories

Today it is difficult to talk about a comprehensive field of research given that various disciplines have developed quite different conceptualizations of communication with respect to the questions that have been relevant in

their respective communities (R. T. Craig, 1999). As early as 1970 Dance claimed that definitions and conceptualizations had become so many and diverse that it is more appropriate to theorize communication as a family of related concepts rather than a single concept (Dance, 1970). Investigating definitions of communication in different publications, Dance identified three components to be particularly critical for the conceptual differentiation: 1) level of observation, 2) intentionality and 3) normative-judgment. The level of observation is about what is chosen to be included in the communicative field. If all processes that link living creatures were to be included, this would entail a rather broad perspective, compared to focusing only on human communication. Should a definition focus on human communication, there are still levels of observation where some definitions would include, for example, individuals' choice of clothes while others would single out spoken or written language. On the level of intentionality an act is defined as communicational based on whether it was intentional or not, a differentiation that significantly narrows or broadens the scope of communication. Intention is in itself a problematic concept with several different definitions (Blakemore & Decety, 2001; Kaldis, 2013; Wegner, 2002). On the normative-judgment level, some definitions include an idea of the effectiveness, success or accuracy of communication, while other definitions take no notice in whether a message has been understood or even received (Dance, 1970; Littlejohn & Foss, 2011).

In a highly influential article, Craig (1999) proposed a constitutive metamodel of communication theory in which he argues that, as the primary means by which humans experience life, communication also shapes experience and thus constitutes reality. The constitutive metamodel of communication does not explain what communication really is, but rather implies that communication can be, and is, constituted symbolically in different traditions. Craig suggested that communication theory as a field could be thought of as having evolved along the lines of seven distinct traditions: the rhetorical, the semiotic, the phenomenological, the cybernetic, the sociopsychological, the sociocultural and the critical. Craig later added an eighth tradition, the pragmatic (R. T. Craig, 2007). These traditions all have variations, and Craig is the first to acknowledge considerable overlap between different branches, but nonetheless, he suggests that they differ in their assumptions about the nature of communication as well as on the different questions and problems to which they attend. These seven traditions are used in the first study of this thesis, where they helped shape the

analytical questions. Each of these seven traditions is presented briefly in the first article *The Logic of Communication in Competency Frameworks for Mathematics*. The theory of communication that has been the most important for the thesis as a whole is the social semiotic theory, which has developed from the semiotic tradition. This theory is presented below.

A social semiotic theory of communication

The social semiotic theory is central in the last two studies of this thesis, and, given that it shares many ideas with discourse analytic research, it is central also in the thesis as a whole. Four concepts from social semiotics that are important to the overall understanding of the thesis are presented: signs, agency, mode and semiotic resource. A particular branch of social semiotics is that of multimodality which, rather than being a theory, constitutes a domain of inquiry (Kress, 2010b). Multimodal approaches to research are based on a social semiotic understanding of communication and they operate with the assumption that communication involves more than just language. Language is assumed to be one of several modes for meaning making, and multimodal approaches acknowledge that all modes have the potential to contribute to meaning (Jewitt, 2011b). In the third and fourth study a multimodal approach to analysis of student texts provided an important base, given that mathematics is a multimodal discipline (O'Halloran, 2005).

Signs

In social semiotics the focus is on the *making* of signs, as opposed to the *use* of signs. Social semiotics adopts the notion that signs are always newly made and, as such, they are always motivated by the interest of the sign-maker. Interest arises through the history of the sign-maker, whether cultural, social or psychological, and is tied to the specific context (Kress & Van Leeuwen, 2006). This perspective represents a break with the tradition in which meaning is "the real and objective description of the intrinsic properties of objects or states of affairs" (Radford, 2006, p. 39) and is thus not subject to negotiation. To produce a sign, a sign-maker uses the most appropriate representational mode available. A semiotic perspective on mathematical activity offers a conceptualization of the teaching and learning mathematics as driven by a focus on signs and sign production (Ernest, 2006). In mathematics education, a semiotic approach has the potential to integrate the individual and social dimensions of mathematical activity while also transcending the traditional subjective-objective dichot-

omy, since signs are intersubjective. Signs provide a basis for subjective meaning making as well as the basis for shared human knowledge, which, Ernest argues, is generally taken to be objective knowledge (Ernest, 2006).

Agency

Another important aspect of social semiotics that is essential to the thesis is the concept of agency. Social semiotics is concerned with *design* in communication, a concept which refers to the semiotic work needed by individuals in order for them to realize their interests in communication (Kress, 2010b). Social semiotics thus places agency in the hands of the individual. Social semiotics, and, in particular, multimodality, assumes that people communicate—create meaning—by selecting and combining different modes (Jewitt, 2011b). Communication involves semiotic work, and, thus, constitutes a principled engagement with and in the realization of meaning (Mavers, 2009). This applies also to children and young students, who, according to Mavers, invest serious semiotic work even in seemingly unremarkable texts. When young students are regarded as social agents who do semiotic work, this shifts the focus from viewing their communication as simple, irrational and undeveloped to viewing their work as something to be taken seriously (Mavers, 2010).

Mode

An important social semiotic concept is mode. A mode, Kress (2011) argues, is a resource for meaning making, and examples include image, writing, speech, music and gestures. Modes are forms of representation that are shaped by history and culture and, as such, they offer different potentials for meaning making. Considering all modes to be resources for meaning making raises questions about their relation to language. Modes such as intonation, gaze and facial expressions could either be seen to duplicate and enhance what is already made clear in language or to contribute to full meanings that may alter, beyond recognition, what is expressed in language. In the latter view, language loses its dominant role in communication and becomes one of several modes for meaning making (Kress, 2011). What can be communicated through a certain mode is specific and partial in a particular culture. Different cultures also have different modal preferences. Western cultures have, for example, preferred writing over image for most public communication for a very long time. The reach of a certain mode also varies from one culture to another, as that which is accomplished through speech in one culture may be better accomplished

by gestures in another. For this reason, meanings are not easily translated between the same modes in two different cultures. What counts as a mode can be decided socially or formally (Kress, 2011). Socially a mode is a mode if the community that uses it considers it a mode. In formal social semiotics there are specific requirements for a mode. A mode has to fulfil three functions: the ideational, the interpersonal and the textual function. To fulfil the ideational function a mode has to be able to represent states, actions and events, or 'what goes on' in the world. To fulfil the interpersonal function it has to be able to position people in relation to each other, and to fulfil the textual function the mode has to be able to form connections with other signs to produce coherent text. Kress (2011) exemplifies these functions by asking: Is layout a mode? The questions that need to be asked in order to confirm that layout is, formally, a mode are: can layout represent what goes on in the world, can it represent the relations of the interlocutors of communication and can it represent these meanings as texts which are coherent internally and cohere with their environment? Kress argues that layout has the potential to meet all three criteria. Layout affects the textual coherence by organizing elements according to a certain internal logic, it represents states or events by arranging elements so as to show causation, connection or relation and it is concerned with its relation to readers through expectations of shared conventions such as reading direction.

Semiotic resource

The term *semiotic resource* originates in the work of Halliday and refers to a resource for meaning making. Van Leeuwen defines the term as

> the actions and artefacts we use to communicate, whether they are produced physiologically – with our vocal apparatus; with the muscles we use to create facial expressions and gestures, etc. – or by means of technologies – with pen, ink and paper; with computer hardware and software; with fabrics, scissors and sewing machines, etc. (Van Leeuwen, 2005, p. 3)

In social semiotics the term semiotic resource can be seen as having has replaced the traditional term *sign*. These resources have a theoretical potential for assisting us in meaning making; this potential is based on the past as well as the potential uses of the resource in communication. Resources also have actual semiotic potential based on past uses that an actor in communication is familiar with and considers relevant, as well as

the potential uses that are considered apt for the specific interests of the actor. Almost everything that is communicated can be communicated in different ways so as to express a particular meaning. Semiotic resources are, thus, not restricted to speech, writing and drawing (Van Leeuwen, 2005). Like modes, semiotic resources are socially made. They are, therefore, to a certain extent regulated by the social occasions and events in which they are used. This entails some stability without implying that they are fixed (Kress, 2010b). Modes, as discussed above, are examples of semiotic resources.

Writing

Writing as a way of communicating has been an issue for scholars for centuries, and the characteristics of writing that engage contemporary scholars in social research were also discussed by philosophers in ancient Greece. Plato, for example, described writing as a form of communication that can travel in time and space away from its author (Plato, 360 BC). The author may not be available to answer questions about interpretations, nor may she necessarily be consistent in her answers if she is available. Time may change her ideas, as she will undoubtedly have had new experiences, all of which have the potential to alter any ideas she might have held as true at one point. She may also have trouble recalling the ideas that went into her text to begin with. Thus, the only way a text can answer questions about its meaning is to have a reader interpret it. A reader however, can only interpret the text she reads and not the text the author wrote. This was one of the reasons Plato preferred dialogue. A written piece of text, being separated in time and space from its author, has to do all its communicative work at once and an author must reflect on the possible readings of different interpreters (Gee, 2008).

That there are several possible readings of a text is acknowledged in many theories on communication and language, such as discourse theory and semiotics (Potter, 2010; Van Leeuwen, 2005). The number of possible ways to read a text, however, is restricted. Many human experiences are common, which leads to shared frames of reference. In particular cultural spheres people understand each other because they share a particular frame of reference, they are using language in the same way and they can therefore 'know' things with some certainty (Öhman, 2006).

> ´We are quite sure of it´ does not mean just that every single person is certain of it, but that we belong to a community which is bound together by science and education. (Wittgenstein, 1969-1975, #298)

This can also be described, with reference to Wittgenstein, as a shared language game. Our knowledge of a language game assists us in our understanding, but it also constrains us. Particular forms of writing are appropriate to the different social contexts in which they are created and read. Problems with laying down criteria for "excellent" mathematical writing include the fact that that any assessment of such writing depends on the knowledge and experience of the reader (Steenrod, Halmos, Schiffer, & Dieudonné, 1973).

Writing can be used to document activity, and this is common in school mathematics. If one defines mathematical writing as *the documentation of mathematical activity*, it may be fair to say that the product of such writing is a mathematical text. As with any text, one characteristic of a mathematical text is that it is always separated from the actual activity. The text is a product and, as such, it succeeds the activity. It is separated in time, and often in space, from the mathematical activity it was created to describe (Love, 1988). Love argues that when a mathematical text is used to infer the processes preceding its creation, there is also an assumption that the description implies that a certain process must, or should, have happened. Processes are then categorized and reified as strategies that come to "appear to exist as things, although they do not necessarily exist at the level of consciousness of the individual problem solver" (Love, 1988, p. 259). This is an unavoidable problem, since the creation of categories and concepts to describe activity is inherent in our ways of thinking. The best we can do, Love argues, is to "be aware that we are creating such concepts rather than uncovering existing ones" (1988, p. 260).

The particular characteristics of writing mentioned above, its separation in time and space from its creator and from the activity it was supposed to document, the various number of readings that are possible and the dependence on the experience and knowledge of the reader are all important points of departure for this thesis.

Representation

Akin to issues associated with writing are issues of representation. Representation has been described as having a dual nature, as the term refers to

both the process of representing something and the product, the representation itself (Goldin, 2014; Stylianou, 2013). A representation is something that re-presents, encodes, stands for or embodies a meaning or idea (Duval, 2006; Goldin, 2014). Goldin differentiates between external representations, which are external to the producer and, as such, accessible to others to observe, interpret and manipulate, and internal representations, which are mental or cognitive constructs. This idea is criticized by, among others, Radford (2014), who argue that the divide between the sensual and the conceptual is unnecessary. Radford proposes a view of embodied cognition in which concepts, in order to become objects of thought, have to be set in motion, to be actualized. The actualization is multimodal and sensuous and, thus, material.

Representations in mathematics can be conventional, as part of an institutional mathematical discourse, or personal and idiosyncratic, and students' representations can be considered to lie on a continuum between conventional and idiosyncratic (Goldin, 2014; Smith, 2003).

Although mathematical representations do not have to be written, school mathematics often involves reporting mathematical work in written form, producing a mathematical text. The focus in this thesis is on mathematical texts that are created and designed by young students in school mathematics. These texts are thought to contain students' representations even if the term as such is not used beyond this section. Instead terminology from the field of social semiotics, such as mode and semiotic resources, will be used.

Writing as a sociolinguistic object

Writing, although it constitutes a complex sign, is often viewed as a single object, which, in terms of communicability, is evaluated in its totality (Blommaert, 2013). When such a composite judgment is disassembled, it is possible to distinguish a range of components of writing. When viewed this way writing can be said to require a range of different resources. These resources are infrastructural, graphic, linguistic, semantic, pragmatic, meta-pragmatic, social and cultural. Together they form a submolecular structure of writing, and because they may be distributed differently, individuals each have specific configurations of resources. For writing to be evaluated positively, each of the different components of writing needs to be organized according to specific norms (Blommaert, 2013).

All forms of writing require some form of material infrastructure, be it pen and paper, a computer, an Internet connection, a whiteboard or a mobile phone. That the medium used to communicate is important to the interpretation of the message is recognized in media studies. Some will even go so far as to say that it is more important than the message itself because it precedes it (Strate, 2008). Blommaert (2013) argues that the infrastructure of writing is very often overlooked as an issue because it is taken for granted.

Writing also involves the capacity to draw, design and organize other visual symbols in specific ways. Design aspects of writing include the use of punctuation marks, paragraphs, sections and chapters, or the use of other text-shaping resources for spatial organization as well as highlighting and emphasizing specific elements in a text. The graphic shape of a text is often used in identifying the particular genre in which the text is written (Blommaert, 2013).

The language variety used in writing needs to be organized according to the norms of a particular genre in order to achieve adequacy. Within each genre, there are strong expectations regarding the ways linguistic resources are used. Examples of linguistic resources that are particular to mathematical texts are, apart from the technical lexis, grammatical devices such as the grammatical metaphor, the relational clause and the use of resources from the nominal group (Veel, 2005). Other linguistic aspects of mathematic language include the multiple semiotic systems that bring together symbolic representations and visual images. The way in which these different modes are combined to create meaning affects the interpretation of a text's adequacy (Schleppegrell, 2007).

Just like speakers, writers need to submit specific norms and to draw from a common set of resources from the particular discourse. Meaning needs to be made understandable to an interpreter through the selection made by a writer from the available options that constitute the meaning potential (Halliday, 1978). Different elements of a text are actualizations of this meaning potential and they represent semantic choice. What is drawn or written in a text represents an assemblage of choices where every choice is relational. These choices are made "with reference to, and have consequences for, other choices: with a view to those already made, relative to those being made and in anticipation of those yet to be made" (Mavers,

2010, p. 8). Meanings are made with available resources that are regulated by culture that is, in turn, sustained by convention and power. Even violations of the norm are norm-governed in the sense that their meaning is dependent on their patterns of language usage (Blommaert, 2013).

Every form of language usage is made sense of from within social and cultural conventions. Conventions are social and cultural patterns of normative organization which develop relatively slowly. These patterns, which can be termed 'genre' or 'register', are patterns of recognizability. They are recognized 'as something'—a friendly text message which demands instant response, for example—or as a mathematical text. The basis for such recognition are connections between specific formal features and contextual ones (Blommaert, 2013). In choosing the resources deemed most apt for a particular text in a particular framing, text-makers attend to a whole range of questions regarding the what, how, for whom and why of the immediate situation, as well as the broader cultural context (Kress, 2010b).

Communicational competence

The concept of communicative competence was introduced by Hymes in 1972 to emphasize that the knowledge of grammatical rules is not sufficient for speaking a language or communicating, and it rests on the notion that in particular situations, not all things that can be said and done are equally competent (Rickheit, Strohner, & Vorwerg, 2008). This idea is fundamental to the rhetorical tradition in communication theory in which communication is conceptualized as an art—an ability that can be cultivated and mastered (R. T. Craig & Muller, 2007; Littlejohn & Foss, 2011). The rhetorical tradition has a long and rich history which traces its roots back to ancient Greece and Rome. In Athens rhetoric was practiced by "rhetors" – persuasive speakers – who engaged in the artistic practice of discourse. In his dialogue *Gorgias* (380 B.C) Plato addressed what he saw as the problem of rhetoric; that rather than being a true art, rhetoric operates through flattery, and there is no interest in pursuing the truth. For this reason Plato viewed rhetoric as a suspect undertaking, something from which scholars should be discouraged. Instead, they should pursue the art of dialectic, which, according to Plato, engages in the pursuit of truth (R. T. Craig & Muller, 2007). One traditional element of rhetoric is *persuasion*, which can be interpreted as representing a conscious effort to

change others and, in doing so, gain control over them (Foss & Griffin, 1995).

The element of persuasion is echoed in Hymes' ideas on communicative competence, but it is just one of several perspectives. There are fundamental human needs that are facilitated by our making ourselves understood. Hymes argues that in order to understand and produce messages, we rely not only on our knowledge of the language at several levels (or nonverbal means to express ideas), general knowledge about the world, cultural schemata and represented constraints, specific situation models, and representations of our own mental and physical states, goals and intentions, but also assumptions about the other person(s) involved in the communication and about their goals, intentions, feelings, attitudes, opinions and knowledge (Hymes, 1972). Communicational competence thus comprises a number of fundamental interaction skills, such as nonverbal communication skills, discourse and conversation skills, message production and reception skills, and impression management skills. Functional skills such as informing, explaining, arguing and persuasion can also be included (Rickheit et al., 2008). Communication can be defined in different ways. If viewed as interaction enacted to reach a certain goal, the concept of effectiveness becomes central: how do I communicate in order to reach my goal as effectively as possible? For a student of mathematics such a goal may be to make a teacher understand and appreciate a certain explanation. If communication is viewed as social interaction which creates and sustains existing sociocultural norms, then the concept of discourse becomes central: what type of communication is appropriate in this particular discourse? For a student of mathematics, this involves identifying and mastering the particular norms for communication that are associated with school mathematics. The latter example is parallel to Wittgenstein's ideas on language games, in which a particular social situation is associated with a particular language game or, conversely, a particular language game is associated with a particular social situation (Wittgenstein, 1953/1986). A participant who initially is unfamiliar with the rules of the game stands out as an outsider until she has become familiar enough with the situation and the language game. Wittgenstein argued that language games are not fixed; new games come into existence and old ones are forgotten. If, at a certain point, one is proficient in a certain language game, there is no guarantee that the rules of the game will not develop into something one is no longer familiar with. Being familiar with the various language games of your mathe-

matics classroom at the age of 9 does not mean one is equally competent in the games at a later age. With this view communicative competence is judged based on its appropriateness according to the social factors in a given situation.

What should be regarded as competent communication is, thus, dependent on how communication is defined as well as on the time span. The dual criteria of effectiveness and appropriateness provides a framework that most competence theorists accept as generally viable (Rickheit et al., 2008). An individual may be judged as having communicative competence if her communication is effective, in the sense that whatever action she intended to achieve is, in fact, achieved and appropriate in the sense that it follows the rules of the particular language game involved.

Mathematical communication

This section deals with theoretical assumptions regarding the relationship between mathematics and communication. This relationship plays an important role in the thesis as a whole. The section also deals with mathematical language and mathematical writing by presenting definitions and ideas that underpin the studies as well as the thesis.

In mathematics education, language and communication play a special role because the concepts used in mathematics are not tangible and therefore cannot be accessed materially. From this claim, two different theoretical perspectives on the relationship between communication and mathematics have developed (Morgan et al., 2014).

The first perspective views mathematical objects as having an existence independent of our different ways of experiencing them. Such a position recognizes that representations, such as signs, words, symbols, expressions or drawings, are our only way of accessing mathematical objects, but it also cautions us not to confuse these objects with their semiotic representations (Duval, 2006; Otte, 2006). Bradford and Brown (2005) alludes to this with a reference to René Magritte's famous painting of a smoker's pipe with the caption *Ceci n'est pas une pipe* which translates to *this is not a pipe*. The painting captures the complex relationship between signifiers (the painting) and that which is signified (a pipe). Just like the caption says, it is not a pipe; it is a painting of a pipe.

To Platonists, mathematical objects exist outside of time and space and independent of human thinking, while other schools of thought view mathematical objects as having arisen from structures, patterns, and regularities either in the physical world or in and of human actions and mental operations (Dörfler, 2002). The common denominator of these epistemological positions is a view of mathematical objects as having some kind of referent outside of the text or the discourse and that mathematics essentially deals with these referents and their properties (Dörfler, 2002). Dörfler argues for the possibility that this is, in fact, a widespread tacit belief among many mathematicians, leading them to adopt a perspective in which mathematical objects are already there just waiting to be grasped by students. This perspective separates mathematics and mathematical knowledge from communication, which leads to a description of learning mathematics as a process mediated by communication (Lampert & Cobb, 2003; Morgan et al., 2014). When learning mathematics is assumed to involve the process of acquiring knowledge and understanding *through* communication, research tends to focus on investigating classroom discourse and interaction along with achievement outcomes (Lampert & Cobb, 2003). One problem with this line of research is that it does not clarify whether it is students' understandings or their ability to communicate that is enhanced by the investigated features of instruction or interaction.

The second perspective rejects this dualist separation of mathematical objects and communication and argues instead that mathematics is a form of discourse and that mathematical objects only exist discursively. With this perspective, a mathematical object can be defined as the sum of the ways of communicating about it (Sfard, 2008). Learning mathematics is then described as a process in which a learner develops mathematical ways of communicating (Morgan et al., 2014). It can also be described as the appropriation of what Wittgenstein referred to as a language game (Dörfler, 2002). With this perspective *language* and *content* are not seen as two distinct accounts, one relating to language, and the other to content, but instead are seen as inseparable (Barwell, 2005b). Research in this perspective sees communication as an aspect of mathematical activity and views the classroom as a community of learners in which the teacher supports the development of a productive mathematical discourse (Lampert & Cobb, 2003).

The two perspectives described above have consequences for the way mathematics is taught, but Lampert and Cobb argue that learning to communicate cannot be cleanly separated from communication as a means by which students develop mathematical understandings (Lampert & Cobb, 2003). Language and communication are taught and learnt in the mathematics classroom, but language and communication are also the primary means by which this teaching and learning is accomplished. This dual nature of writing as a means and as a goal in mathematics plays a central role in the so-called genre wars (see Pimm & Wagner, 2003), in which the two possible positions emphasize, in turns, a) paying explicit attention to the form, to explicitly teach students to write in and understand different genres, or b) the opposite, leaving the form and instead focusing on authorship and creativity in the hope that successful communication eventually will become inculcated through the students' schooling (Pimm & Wagner, 2003; Solomon & O'Neill, 1998). There are parallels between the dual nature of communication in the learning of mathematics and Sfard's two metaphors for learning, the acquisition-metaphor and the participation-metaphor (Sfard, 1998). Just as it is not possible to separate communication as a means and a goal, these metaphors are not mutually exclusive, and Sfard argues that too great a devotion to one particular metaphor can lead to theoretical distortions as well as undesirable practical consequences. Researchers, Sfard argues, need to accept a reality constructed from a variety of metaphors that are only ever good enough to fit small areas (Sfard, 1998). In this thesis, like Lampert and Cobb and Sfard, respectively, I argue for a view of communication as something that is a means and a goal at the same time. Students learn mathematics through communication and they also learn to communicate. By acknowledging that mathematical communication involves more than the use of formal mathematical expressions, the teaching of communicational skills will include aspects of communication such as what, how, for whom and why students communicate, along with questions regarding what signifies good communication.

Mathematical language

The concept of mathematical language can refer to a number of entities. In mathematics education language has been taken to mean: 1) the method of human communication, spoken, written or any other non-verbal method of expression or communication, consisting of the use of words in a structured and conventional way; 2) a system of communication used by a

particular country or community; and 3) the phraseology and vocabulary of a particular profession, domain or group (Morgan et al., 2014).

The third definition is associated with what Halliday (1978) called *register*, "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1978, p. 195). The mathematics register includes words that are specific for mathematical communication, but it also includes everyday words which are taken to mean something specific in a mathematical context. Halliday does not include formal mathematical symbolism in his description of the mathematics register. Mathematics language, Halliday argues, is a special form of natural language which is used in mathematics.

If mathematical language is taken to mean 2) a system of communication used by the mathematics community, it is possible to include mathematical notation, but then defining the mathematics community becomes problematic. Does such a community include mathematics teachers and students on different levels of the school system?

If mathematical language is taken to mean 1) a method of human communication consisting of the use of words in a structured and conventional way, it can be argued that this is a special language which differs from that of most other disciplines. Mathematical language is defined by its multi-semiotic nature in which both linguistic, visual and symbolic semiotic systems contribute to meaning in different ways (O'Halloran, 1998, 2005). The mathematical language also seems to have endless applications (Veel, 2005). As a language it displays particular features such as a highly technical vocabulary, particular grammatical patterning, such as dense noun phrases and implicit logical relationships (Dunsmuir et al., 2015; Schleppegrell, 2007).

The mathematical language that is taken to be the norm today can be said to represent the result of an increased standardization and a narrowing of linguistic options in the describing of mathematical operations throughout the last two centuries (Solomon & O'Neill, 1998). Because the concepts constructed in mathematics are often difficult to articulate in ordinary language, mathematic symbolism has developed to express meanings that go beyond what ordinary language can express (Schleppegrell, 2007).

Mathematical activity takes place in social contexts, but the most essential characteristic of any mathematical creation is its universality and independence of context. These are the two features to which mathematics owes its strength as communicational tool (Sfard, Nesher, Streefland, Cobb, & Mason, 1998).

Mathematics can be viewed as a precise language that is used to describe different aspects of the world with as little ambiguity as possible. Mathematical vocabulary, possibly more than in any other discipline, is seen as being precisely defined (Barwell, 2005a; Morgan, 2005). Barwell argues that it is common to ascribe ambiguity in mathematical expression to poor use of language on behalf of the communicator rather than to uncertainty in mathematical ideas. In this view, the introduction of new words involves explaining and clarifying their meanings, which directs students to a particular understanding of words ready for use (Barwell, 2005a; Brown, 1997). An elementary version of this perspective is common in school mathematics where there are only right and wrong answers to everything, along with clear-cut methods for finding these answers, and, subsequently, there is no place for opinion and nothing to discuss. "While there might be open problems at the frontiers of mathematics, it is all sorted out and written down at the school level." (Bell & Bell, 1985, p. 47). This view of the language of mathematics has possibly affected how all communication in mathematics is viewed, and Morgan has noted that

> (t)he whole process of assessment appears to be based on an assumption that writing is 'transparent'. In other words, the written (or oral) text is assumed to convey the intentions of the author, without distortion or alteration into the mind of the reader…Such an assumption is based on a 'common sense' or a naïve transmission view of the nature of communication (Morgan, 1998, p. 197).

Morgan (2000) argues that although it is generally recognized that students can interpret teachers' statements in a number of different ways, it seems as if this notion rarely is extended to how teachers interpret what students say or write. She suggests that a more consistent epistemology would reject the simple correspondence between a piece of text and the meanings that different readers construct. Such an epistemology suggests that the meanings will depend on the resources and experiences of each individual reader rather than the text itself. With this view it seems clear that mathematical expressions are not transparent, but rather that their

meanings will depend both on how the individuals perceive their task and on their familiarity with this type of expressions (Brown, 1997; Morgan & Watson, 2002). Meaning becomes subjective, situated and in a constant state of flux. This does not mean, however, that meaning can be anything. Individuals do not construct meaning in a vacuum. Instead they share experiences of social interaction and they develop their idea of meaning in relation to these experiences (Öhman, 2008). As described earlier Wittgenstein suggested that this could be looked at as a game—a language game. From this perspective it is possible to say that the idea of a mathematical object like vector does not have an intrinsic meaning; neither does the word 'vector', nor an arrow on a page. The meaning of these things is connected to our engagement with them, which is a social process, involving people (Barwell, 2007). Where mathematical meaning is dependent on individual interpretation, the possibility of ambiguity is ever-present (Barwell, 2005a).

Language is complex, and drawing the line between what is mathematical language and what is not is difficult, as it depends on several things (Barwell, 2007). To complicate things further, there is a difference between the language of mathematicians and the language used in the mathematics classroom (Morgan et al., 2014). When language is used in the mathematics classroom, it constitutes a goal and a means, and these two functions cannot be separated (Lampert & Cobb, 2003). People can talk mathematically, they can talk about mathematics and they can talk about talking about mathematics. Different levels of conversation in which mathematical language is intertwined with natural language that is part of our everyday speech are maintained concurrently (Forman, 1996; Sfard et al., 1998). Natural language also encompasses non-verbal modes of communication such as gestures, intonation, facial expressions and body language.

Mathematical writing in school

The integration of mathematical language and natural, everyday language is also visible in mathematical writing. It can be argued that written mathematical presentation consists of two parts: the formal or logical structure that consists of definitions, theorems, and proofs, and the complementary informal writing that consists of motivations, analogies, examples, and meta-mathematical explanations (Steenrod et al., 1973). In the first part the formal language of mathematics is used, but in the second part the

natural language which is part of everyday communication, is used, including images (Sfard et al., 1998). This applies to all mathematical writing, but may be seen as particularly important in school mathematics, where students have limited access to the formal language. It can be argued that natural language is limited when it comes to describing mathematical notions, and mathematical communication is partly the ability to formulate mathematical expressions in such a way that will distinguish between several possible meanings. One example is the use of parentheses. By agreeing about the role of parentheses in formal notation, mathematicians are able to make distinctions that in natural language would require access to several resources for meaning making, such as intonation, which is difficult to achieve in writing (Sfard et al., 1998). Natural language, on the other hand, owes its strengths to its flexibility, which offers a nearly endless number of ways to convey meaning through multiple modes. Order, the use of space, grouping, joining with lines, underlining, circling, pointing with arrows and sequencing are all examples of resources that contribute to the meaning of a multimodal text. The meanings conveyed through such resources may be subtle, but they play a part in the meaning making of any reader.

Mathematical literacy

Writing skills can be thought of as part of a student's overall mathematical literacy. This concept is explored below, starting with the question of what literacy is and how this concept has evolved.

The concept of literacy

Assessment in mathematics has many concerns, of which perhaps the most important one is: what is it that is being assessed? This issue has been dealt with and given many names throughout the history of mathematics education. In this section the *thing* that is, or rather, should be, assessed in mathematics is referred to as mathematical literacy. The term *literacy* was originally associated with the technical, and, as such, neutral, ability to read and write, but has since evolved and in contemporary disciplines such as new literacy studies, in which the term is understood as having an ideologically situated nature. Hence, there are a number of different ways to read and write (Street, 2005). If *literacy* is taken to mean 'being able to read', the history of literacy is as old as writing itself (Gee, 2008). Gee claims that literacy from the very beginning has been "known" to be something immensely positive to people and societies. Countries with high

literacy rates are widely believed to be better developed, and more modern and literate people more intelligent, more modern and even more moral. This, according to Gee, is all a myth, since there is little historical evidence for such claims. The reason for this lack of historical evidence, Gee explains, "is because the role of literacy is always much more complex and contradictory, and more deeply intertwined with other factors, than the literacy myth allows" (Gee, 2008, p. 47). To the discussion on the history and development of literacy, Gee introduces Plato's dilemma. As discussed above, Plato preferred dialogue to written text. Writing, Plato said, cannot defend itself or stand up to questioning; it only repeats what has already been said. Plato wanted to ensure that the voice behind the text could dialogically respond to questions. Unless the "voice" can respond, there is a risk that interpreters, who might be ignorant or lazy, come up with the "wrong" interpretation. The "voice behind the text" must be privileged and there has to be a possibility of ruling out some interpretations. But the ruling out of an interpretation is a political act, an act of power, and, as such, it could lead to authoritarianism, which in turn destroys dialogue. This is the core in Plato's dilemma; one that Gee claims is hard to get around: "Literacy seems to require some authority that determines what interpretations count (or all count and there is no meaning), but that authority can be self-interested and kill dialogue" (Gee, 2008, p. 53). These arguments are all grounded in the assumption that there are different ways to "read" a text. This is a move away from thinking of reading as an entirely cognitive endeavor concerned with the mechanical decoding of letters. To a majority of modern linguists terms like reading, text and literacy, as pointed out above, are situated in society rather than in the individual. Social linguists within the tradition of new literacy studies (Gee, 2008; Street, 2005) argue that language is always language-in-context. This, Gee claims, is a deconstruction of old ideas of what literacy does to people. Scholars in fields such as linguistics, anthropology and psychology use different methods to show that literacy has different effects in different social contexts. This leads to difficulties for anyone wishing to maintain a of view literacy as being the natural characteristic of "modern and sophisticated" societies or people. Since there are endless different social contexts, literacy cannot be seen as a singular thing but "a plural set of social practices: literacies" (Gee, 2008, p. 63).

Knowing mathematics

When it comes to mathematical literacy, the arguments are similar as the ones presented above but not entirely the same. Just as in discussions on literacy, scholars in the field of mathematics education have pointed out how difficult it is to provide a simple and comprehensive definition of mathematical literacy (Jablonka, 2003; Kanes, 2002; Wedege, 1999, 2010). Different scholars give slightly different reasons for this, but their arguments are centered on the functional aspect of mathematics. Jablonka (2003) argues that since mathematical literacy is about an individual's ability to use and apply mathematical knowledge, it has to "be conceived of in functional terms as applicable to the situations in which this knowledge is to be used"(Jablonka, 2003, p. 73). Any kind of literacy is always literacy about something, be it labor demands, the demands of society or something else. Jablonka identifies these 'somethings' as developing mathematical literacy for developing human capital, cultural identity, social change, environmental awareness and evaluating mathematics. Like Jablonka, Wedege (2010) and Kanes (2002), albeit in different ways, touch on the political dimension of mathematical literacy and its implications for social issues like equity. They all stress the fact that mathematics is "about" something and that it is the "use of" mathematics that one is concerned with when talking about mathematical literacy.

Definitions of mathematical literacy

In spite of the problems associated with defining "a" mathematical literacy, several different definitions have been suggested at different times under the names of numeracy, mathematical proficiency, mathemacy, matheracy and quantitative literacy, to name a few (Wedege, 2010). Kilpatrick (2001) uses the term mathematical proficiency when he relates to the 1995 American initiative that ultimately resulted in a definition of 'successful mathematics learning'. In this definition, five strands of mathematical proficiency are identified: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. This definition deals with the problem of mathematics being contextual by not defining any context and instead using general terms that could be applied to (supposedly) any situation.

A project similar to the American initiative is the Danish KOM project (Niss, 2003; Niss & Højgaard Jensen, 2002). The core of the KOM project was to carry out an analysis (Blomhøj & Højgaard Jensen, 2007) of

mathematical competency defined as someone's insightful readiness to act in response to a certain kind of mathematical challenge in a given situation, and then identify, explicitly formulate and exemplify a set of mathematical competencies that can be agreed upon as independent dimensions in the spanning of mathematical competence. The KOM project ultimately resulted in a suggestion of eight different competencies: reasoning competence, modelling competence, problem tackling competence, mathematical thinking competence, representing competence, symbol and formalism competence, communicating competence and aids and tool competence. As is obvious from the description of the aim of the project, the contextual nature of mathematics competence is dealt with much in the same manner as in the American initiative, i.e. by using supposedly general terms.

The last example of a definition put forward here is that of the PISA Framework by OECD. It is different from those described above since it supposedly uses an "innovative 'literacy' concept" (OECD, 2013, p. 16) which is concerned with the capacity of students to "apply knowledge and skills" as well as to "analyse, reason and communicate effectively" while they "pose, interpret and solve problems in a variety of situations"(OECD, 2013, p. 16) The PISA Framework defines mathematical literacy as

> [...] an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make well-founded judgments and decisions needed by constructive, engaged and reflective citizens (OECD, 2013, p. 25).

In the PISA Framework, OECD argues that the use of the term mathematical literacy "emphasizes the need to develop students' capacity to use mathematics in context" (OECD, 2013, p. 25) Jablonka (2003) argues that this kind of broad definition of mathematical literacy fails to acknowledge that mathematics is culture-bound as well as value-driven. She argues that it is connected to students learning how to think, but it does it does not deal with learning what to think about. The PISA definition from 2009 claimed to be consistent with the ideas of sociolinguists like Gee and Street, but the mere offering of a single definition of a single mathematical literacy seems to contradict one of their main ideas, that

there are as many different literacies as there are social practices. Anyone offering a definition inevitably has to deal with Plato's dilemma: either you present a definition that accounts for all the different mathematical and social practices, and by that you have said nothing, or you rule out some practices and by doing so you have excluded certain groups in society.

Frameworks for mathematical literacy

The different definitions presented above have resulted in different frameworks in which the definitions are presented and developed. *The Strands of Mathematical Proficiency* (Kilpatrick, Swafford, & Findell, 2001), the *Process Standards and Principles and Standards for School Mathematics* (NCTM, 2000), *The Danish Mathematical Competency Framework* (Niss & Højgaard Jensen, 2002) and *The OECD/ PISA mathematical literacy framework* (OECD, 2013) are examples of constructs that build on the idea that mathematics is an activity and that knowing mathematics is doing mathematics (Boesen et al., 2013). As discussed above, they also build on the assumption that mathematics is a domain in which it is possible to provide a generic set of mathematical practices (see also Säfström, 2013). Säfström notes that the objective of these constructs, presented in non-scientific texts, is "to give advice and recommendations to teachers and curriculum writers to improve mathematics education, more or less based on results of research" (Säfström, 2013, p. 31). She points out, however, that the constructs arose within the field of mathematics education and, as such, attend to the specific characteristics of mathematics. The different aspects of mathematical knowledge or ability, whether presented as *mathematical competencies* (Niss & Højgaard Jensen, 2002; OECD, 2013) or *strands of proficiency* (Kilpatrick et al., 2001) are presented as closely related, over-lapping and interdependent, respectively, rather than mutually exclusive and independent.

Summary

In this section the most important theoretical assumptions that underpin the studies and the thesis have been presented. These include a brief description of the historical development of different conceptualizations of communication which can be seen as an important background to the different ways in which communication has been understood throughout history. A description of contemporary theories on communication was also presented to introduce the various conceptualizations of communica-

tion that have been important for my own overall understanding of communication. These conceptualizations also provided the base from which the analytical questions of the first study were derived. The most important of the different theories of communication has been the social semiotic. This theory foregrounds the social aspect of communication but offers a perspective that can integrate the individual and social dimensions of human interaction. The particular features of writing that are of importance for the thesis are its ability to travel in time and space away from its author, something that makes it a convenient record of various presumed actions, and its potential for inferences regarding the activities, intentions, knowledge and skill of the author. Sociolinguistics is related to social semiotics and the sociolinguistic understanding of writing as consisting of different components has been important for the thesis. The theory behind the concept of communicational competence was also presented. That students can exhibit different communicational competence is a fundamental assumption in the thesis.

The presentation and discussion of communicational competence was followed by a section that dealt with the relationship between mathematics and communication. This relationship plays an important role in the thesis. Mathematical communication involves mathematical language and mathematical writing. One of the most important ideas that underpin the studies as well as the thesis is the idea that mathematical writing includes not only mathematical language, but also natural language, the language used in everyday life. It is important to note that natural language includes all modes of communication with which people make meaning, such as image and layout. This notion provides an essential point of departure in the examination of the mathematical writing of young students.

The ability to communicate in mathematics is part of a student's mathematical literacy. The theory section concluded with a presentation of different definitions of mathematical literacy, along with some frameworks in which these definitions are developed. They all build on the assumption that mathematics is a domain in which it is possible to provide a generic set of mathematical practices. These practices are termed abilities, strands of proficiency or competencies. In the frameworks they are seen as interdependent and overlapping. In practice, however, they need to be distinguishable from other competencies in order to be assessed. A student should be graded with all the different abilities in mind, but it is necessary

for an examiner to be able to differentiate between them. In this case I argue that the *ability to communicate* stands out. Regardless of how communicational ability or competence is defined, it would prove very difficult to assess, or even access, any of the other abilities without communication. The question of whether it is possible to tell the form from the meaning and, by extension, to differentiate the ability to communicate in mathematics from the ability to successfully exercise mathematical activity is an important question that lies at the heart of this thesis.

Methodology

This section elaborates on the methodology of the four studies and discusses the methodological choices in relation to the validity, reliability and generalizability of the results, as well as ethical considerations. The aim of the four studies of the thesis is to reach a deep understanding of writing in school mathematics, through the examination of how the concept of communication is described in authoritative texts, how this writing is viewed by teachers and how it is employed by students.

As in most qualitative studies, the data in all four studies consists of language, written or verbal, that has been used in different situations for different purposes. It can further be argued that all the data is text. In some research disciplines the term 'text' includes verbal as well as written accounts. The data in the four studies, however, consist of written text, given that transcribed interviews are texts once they are transcribed (Silverman, 2011). The studies all adopt a discourse analytic approach where discourse is understood as *language in use* (Wetherell, 2010a) and as *ways of acting, ways of representing and ways of being* (Fairclough, 2003). Defining what one means with discourse may be particularly important in mathematics education research, where it has been noted that the term *discourse* is defined and understood in a number of different ways (Ryve, 2011).

Discourse analytic approaches, in which discourse is conceived as *language in use,* take an interest in the relationship between the language, along with other modes of communication, and the social context in which this language is used. The interest is directed towards the meanings that arise from this relationship (Morgan, 2014). Such approaches see discourse as embedded in practice, involving not only language but also perspectives and knowledge (Gee, 2008; Moschkovich, 2007). When this perspective is adopted, texts and transcripts from a certain practice are seen as carriers of the ideas, presumptions and understandings of that practice. Because language and other modes of communication offer a number of alternatives to expressing a certain idea, the choices that any communicator makes can be used to infer ideas, presumptions and understandings as they are manifested in the way language is used (Machin & Mayr, 2012).

The validity of discourse analytic approaches to the analysis of text builds on the reliability of the interpretation. Meaning is dependent on the frames of reference, the shared set of rules and the social codes of the social context in which communication is enacted (Jaipal-Jamani, 2014). Reliability is thus dependent on a researcher's knowledge in and experience of the *language game* of the particular practice. Other elements that render discourse analysis valid include convergence, agreement and coverage and attention to linguistic detail (Gee, 2011). The methodological decisions of each study will be presented below and issues regarding the validity and reliability of each analysis will be discussed in the method discussion that follows.

This section intends to present and discuss the design of the studies along with the rationale for the methodological choices. The studies are all designed with the aim of examining aspects of students' mathematical writing, but they each focus on different aspects. Below is an overview of the four studies with regard to object of study, data, analysis and important concepts. Following this, the study design processes for each individual study are described. Some of these processes have common denominators, such as multimodal or discourse analytic approaches, across the studies. Even if the studies in this case share a particular view of, for example, language and communication, these views are still described in relation to each study given that they each draw more or less heavily on different concepts within these broader understandings.

Table 1 *An overview of the various objects of study, forms of data, analysis and concepts related to each study.*

	Study 1	Study 2	Study 3	Study 4
Object of study	The communicational logic of competency frameworks in mathematics	Teachers' collective discussions on students' mathematical texts	Students' mathematical writing	Students' digital and analogue mathematical writing
Data	Online versions of competency frameworks	Transcribed group discussions (4h26 m)	519 texts produced by students aged 9-12	28 texts produced by 7 students aged 12
Analysis	Discourse analytic approach to text analysis	Quantitative summative content analysis and a discourse analytic approach	Multimodal discourse analysis and conventional qualitative content analysis	Multimodal discourse analysis
Concepts	Logic	Discourse, collective discussions	Modes, text, resources	Analogue and digital text, resources

The design of the studies

The general design of the studies and the rationale behind the various methodological choices are presented. Following the presentation of the study design the analytic procedure of the studies is presented and discussed.

Study design for the first study

The aim of the first study is to explore and problematize the concept of communication in mathematics education. This is done through the examination of three competency frameworks in mathematics. These are analyzed with the intention of capturing the embedded logic of communication. The analysis is based on an examination of the ways in which communication has been defined in different traditions in communication theory. This provides the inspiration for three analytical questions concerning actors, content and form, which are used to identify the presumptions about communication that emerge from the frameworks.

The frameworks can be seen as examples of authoritative texts (Östman, 1995). Authoritative texts such as curricula, policy documents, frameworks and textbooks, have a certain status within a community or practice. Such texts are also characterized by their reach within the community. According to Östman, authoritative texts comprise the ideology and logic of the practice in which they are written. This would make them ideal objects of study when the aim is to identify their underlying conceptualizations of communication.

To capture different conceptualizations of communication, the study employs the concept of logic as developed by Glynos and Howarth (2007). Glynos and Howarth developed the concept in relation to political science, drawing on Wittgenstein and Laclau respectively. Following Wittgenstein, they describe the logic of a practice as something that comprises the rules and grammar of that particular practice. Following Laclau they also describe the concept of logic as different from discourse, in the sense that it refers not only to the rules of a particular practice, but also to the various conditions that make that practice work. The concept of logic is, thus, designed to capture the presuppositions of a practice or regime. Considering that communication in some form is ubiquitous in nearly every practice, the logic of communication embedded in these practices, can be said to be particularly important for understanding them. The logic of commu-

nication can be thought of as having an impact on every element of educational practices.

The study sets out to examine authoritative texts, and three contemporary competency frameworks for mathematics have been chosen. These are *The PISA 2012 Mathematics Framework (OECD, 2013), The Singapore Mathematics Framework (MES, 2012)* and *The Common Core Standards for Mathematical Practice* (NGACBP, 2010). These frameworks are chosen for several reasons.

Firstly, they explicitly address communication as an ability, practice or skill that is separate from other abilities. This provided an important base from which to explore the ideas about communication that are embedded in descriptions of successful communication. When communication is seen as a separate ability, this ability and its components must be described or explained in some way.

Secondly, the frameworks are identified as influential competency frameworks in mathematics education. The Singapore framework is listed as an example of an influential competency framework in The Encyclopedia of Mathematics Education (Kilpatrick, 2014), whereas the *PISA* framework and the *Common Core* build on, and develop, two earlier frameworks, which are listed in the same encyclopedia. The PISA framework builds on the Danish KOM-project (Niss & Højgaard Jensen, 2002) and Common Core builds on frameworks introduced by The Mathematical Learning Study of the NCTM in the US (Kilpatrick et al., 2001).

The frameworks are also chosen because they are created and designed for different purposes and, consequently, speak to different audiences. The PISA framework is used in an international assessment project where it can be seen as speaking to stakeholders in education on a macro-level. The Singapore framework is part of a national curriculum and applies to teachers in Singapore. The Common Core framework represents an effort to create common standards for school mathematics across the US, which yields a mixed audience consisting of educators on different levels, teachers, students and parents. It would be natural to expect certain differences in the ways in which communication is conceptualized, given that the frameworks are created with these different audiences in mind. Should the analysis, however, fail to show such a difference, and instead uncover a

common communicational logic, then this can be seen as an indication that this logic is particularly pervasive in mathematics education and that it exerts a strong influence on school mathematics in various levels of the education system. The choice of three frameworks that are created for different purposes is, thus, thought of as a way to examine possible differences as well as a way to strengthen the reliability of the result if the analysis indicates that there are few differences.

The frameworks are all available online, and they are all in English, which contributes to their global reach. This reach, along with their relative prominence, is seen as an indication that they influence mathematics teaching and assessment at different levels in mathematics education in a number of countries.

Analysis

The analysis was performed in three steps. The first step involved reading through all of the three competency frameworks twice, using the second read to highlight passages in the texts that were connected with communication. In the second step in the analysis, three analytical questions were used to identify implicit rules in the frameworks in order to examine the presumptions concerning communication that were embedded. The third step of the analysis was based on the results from the three questions of the second step. The relations between the identified rules were examined in order to determine consistent communicational logic(s) embedded in the frameworks.

A prerequisite for an exploration the concept of communication is a thorough knowledge of the various ways in which communication has been defined and conceptualized throughout history. An examination and presentation of seven different traditions in communication theory as described by Craig (1999) serves as an important theoretical base in the study as well as in the thesis. This examination provides opportunities to attend to different aspects of communication as well as to organize and compare the findings. Moreover, it offers an opportunity to make an informed choice concerning what to include in the analysis.

A systematic text analysis concerning communication has to operate with a clear definition of what communication is. Given that a starting point in the analysis of the three frameworks is the identification and highlighting

of passages in the texts that are connected with communication, a working definition is needed. To decide what kind of interactions fall under the category of communication, a working definition is based on the different traditions in communication theory (see article 1). This definition concerns the aspects that Dance (1970) found to be particularly critical for the differentiation of different conceptualizations of communication: 1) level of observation, 2) intentionality and 3) normative-judgment:

1. Communication is defined as human interaction which includes oral and written communication. This definition excludes internal communication, thinking, as this does not involve other humans.
2. Communication is defined as intentional interactions. Only interactions with humans who communicate with the intention to interact with others are included.
3. Communication is defined as interaction which is recognized, as well as acknowledged, by another human, albeit not necessarily understood or required to produce a desired result or response.

For the purpose of the empirical analysis, communication is also thought of as something that has a direction from one human to another and as having an expressive side where communication is directed away from an actor in communication and a receptive side in which the direction is towards this actor.

In the second step of the analysis, the frameworks are explored with the help of three analytical questions. The different traditions in communication theory and their attention to various issues provide inspiration to the analytical questions. Several traditions in communication theory are concerned, albeit in different ways, with the actors in communication. They all assume that communication is inherently social and that the experiences and behaviors of actors in communication will significantly affect any outcome. This notion suggests a question that allows for identification of *actors in communication* other than the student. This identification is not only concerned with the identity of these actors, but also their characteristics. A question regarding who the implied audiences for students' communication are, and what their implied features are, highlights an important aspect of communication. What is the communication about? What are the concepts that are being communicated? These concerns help

shape a question about the content of communication – a *what* question. In the analysis, the content is seen as manifested in the particular use of nouns. For the sake of clarity these nouns are referred to as *messages* despite this term's close relationship to the cybernetic tradition and the sender-receiver model of communication.

Closely connected to the question of content are issues concerning the *form* that communication can take. This concern provides inspiration for an analytical question concerning how the students are, or should be, communicating. Whether the content of communication can, in fact, be distinguished from the language itself is a philosophical question. As described above, content is thought of as manifested in nouns such as justifications, statements, problems, tables or diagrams. These examples, however, also signify a specific form hence the difference between content and form does not seem clear cut. The analysis is therefore directed towards the form through adjectives such as clear, formal and correct, as well as adverbs such as logically, precisely and concisely. These are concerned with form as character or quality, which represents a more abstract level.

The linguistically oriented exploration of the passages that deal with communication is thought of as a way to make the analysis transparent. The concepts involved are multi-faceted and can take on a number of meanings. Tying their meaning to a particular linguistic function contributes to narrowing the scope of their potential meaning.

The result of the linguistic analysis through the three analytical questions is used in order to identify the rules and presumptions that emerge. This final step of the analysis is more qualitative and consequently less transparent. The answers to each analytic question are interpreted as a rule or presumption about communication in mathematics. These claims have to be supported and the reasons for each interpretation presented. Alternative interpretations have to be considered. This is a long process that involves several reassessments of the results. Once the rules are established and validated the relations between the identified rules are examined in order to determine a consistent communicational logic(s) embedded in the frameworks.

Study design for the second study

The aim of the second study is to examine the ways in which mathematics teachers discuss students' mathematical texts. The study takes a discourse analytic approach and the object of study is teachers' collective discussions rather than their individual conceptions. Group interviews were conducted with 19 middle school teachers who were presented with, and asked to discuss, 15 different mathematical texts produced by students in grade four. The transcriptions from the interviews are analyzed through a combination of quantitative summative content analytic and discourse analytic approaches.

The idea was that a group of teachers would be presented with authentic mathematical texts, which were produced by students that they did not know, and asked to interpret and assess them. Such a design is thought of as a way to create a context similar to that of teachers taking part in joint discussions and grading of, for example, national tests. The situation of being faced with a mathematical text whose origin one knows very little about is thought of as a way to force the interviewees to focus on the texts as products. It is assumed that this will lead to a situation in which the interpretations derived to a larger extent from the text than they would if the interviewees had been asked to comment on their own students' written material. When the context in which the teachers normally assess students' texts is replaced with a context devoid of relationships with and knowledge about the students, the teachers' assessment needs to be based entirely on what is actually written rather than on assumptions which are based on familiarity with the writing style of specific students. In Sweden there have been calls for national tests to be assessed and graded externally instead of by the teachers who already know the students. External grading is seen as a way to secure objectivity and fairness. In choosing to use texts from unknown students I have an opportunity to analyse this objectiveness and fairness, even if this is not the main objective of the study.

Prior to the group interviews, eight teachers at six different schools were asked to organize problem-solving activities from which they were to collect mathematical texts. These teachers were presented with two similar problems, which they were instructed to modify or reformulate as they saw fit, and to organize one or several problem-solving activities, possibly in several student groups, in a way that differed as little as possible from

everyday activities in their classrooms. The problems are Diophantine equations with a limited number of solutions. The first problem involved distributing 30 legs on 12 animals, the animals being pigs and hens. The second similar problem involved distributing 36 wheels on 11 vehicles which were either cars or motorcycles. The texts provided by two of these teachers were selected for use in the subsequent interviews, rather than texts from all eight, to provide some control over the variables involved in the contexts in which the texts were created. The two teachers had both collected material from students in grade 4, and from the texts they provided, 15 were selected in order to create a sample that was diverse in relation to students' use of modes, problem-solving strategies and layout. These 15 selected texts, 8 of which represented solutions to the first problem and 7 to the second, were used as the basis for group interviews.

The group interviews were conducted in five groups of 3, 4 or 5 teachers. In four of the groups the interviewees were from the same school and in the fifth group there were two teachers from one school and a third teacher from a different school. At the time of the interview, all 19 interviewees were teaching mathematics along with several other subjects. They are all experienced teachers with between 8 and 40 years in the profession. They were initially identified as good candidates for participation by their school principals and invited to volunteer to participate in the study. Being selected by your principal and asked to participate in a specific research project may be seen as the same as being ordered. One important part of the interviews was therefore to ensure that the teachers were in fact there of their own volition, although there is no way of knowing for certain that that was the case. Due to the fact that the teachers had been selected by their principals, there was no way to ensure diversity or representativity with regard to age, experience, gender or educational background. By coincidence some degree of diversity was achieved in terms of age, experience and educational background, but in terms of gender, the distribution was clearly asymmetrical. This asymmetry is neither representative of the Swedish population nor the Swedish elementary school teacher collective. Given that the aim of the study is to investigate teachers' collective discussions the lack of control over factors such as diversity and representativity are deemed acceptable.

The group interview or focus group originated in commercial market research in the 1940s. Although still an important tool for this type of re-

search, it has also gained popularity in research on social science across a range of disciplines (Wilkinson, 2011). In such research, however, the focus group is often used as a complement to other methods. It can be used to identify research foci, to develop research questions or to clarify or verify research findings. In this study the group interview is used as the primary method of data collection.

An important starting point for the design of the data collection is the belief that people make meaning in different ways in different contexts and that by studying how language is used in a particular context, it is possible to gain an understanding of different practices that are at play (Morgan, 2014; Wetherell, 2010a). The study therefore adopts a discourse analytic approach as a way to deal with the relationship between communication, through language or any other mode, and the context in which the communication takes place. This is also one of the reasons why group discussions rather than individual interviews were chosen. The second reason was the idea that group discussions would provide opportunities to examine the way in which the teacher collective, rather than the individual teachers, interpret ideas and concepts (Frey & Fontana, 1991). The idea that a particular professional collective may share some of the different ways in which they interpret phenomena is important in discourse theory (Gee, 2011; Wetherell, 2010a) as well as in theories of situated learning and communities of practice (Lave & Wenger, 1991; Wenger, 1999). Goodwin (1994) refers to the distinct patterns of what is noticed and what is not, for groups with similar goals and experiences, such as professional groups, as *professional vision*. He provides an example of an archaeologist and a farmer who will see "quite different phenomena in the same patch of dirt" (1994, p. 606). These ideas are also echoed in mathematics education research on teachers' noticing, where the idea is that teachers will notice, and subsequently direct their action towards, that which they find important (Jacobs, Lamb, & Philipp, 2010; Mason, 2011; Sherin, 2011). The group discussions are thought of as a situation from which to extract meanings that are taken as shared, meanings that are contested and meanings that are negotiated. A well-conducted group discussion is thought of as allowing for the identification of different types of meanings.

The interviewees had not seen the material before, and prior to each interview they were given time to familiarize themselves with the 15 texts. They were then asked to discuss the different texts from an assessment

perspective with a particular focus on the way the students communicated their ideas through different modes such as drawings, symbols and words. They were asked to comment on what they noticed in the texts, what they considered important, what they thought was possible to say about the students and their abilities, and finally what they considered to be "good" and "less good" examples of mathematical texts. They were also asked to provide arguments for their reasoning and judgments. Follow-up questions asking the interviewees to elaborate on or justify statements were used when appropriate. Given that I acted as an interviewer in the sense that I did ask questions, there is cause to categorize these meetings as *interviews*. An interviewer is never neutral and the interviews can therefore never be seen as independent of the interviewer as an actor (Holstein & Gubrium, 2011). As the interviews played out, however, these interviews could often rather be characterized as group discussions in the sense that, during the course of all five interviews, the interviewees themselves often acted as drivers of the discussion (Crespo, 2006; Wilkinson, 2011). The interviewees agreed with one another on some occasions but disagreed on others, posed their own questions to the texts, introduced new topics and questioned the topics that were given. They were often interested in, and even fascinated by, their colleagues' arguments.

The discussions among the interviewees, which comprised a total of 4 hours and 26 minutes of recorded material, both audio and video, were transcribed. Transcription of verbal interactions is a process that seldom is foregrounded in empirical studies (Ochs, 1979). Ochs argues that a transcription is a selective process that in many respects reflects theoretical perspectives and goals. Verbal behavior is multimodal in that it includes non-verbal elements such as facial expressions, intonation, gaze and gestures. However, a transcription that includes information on all aspects of verbal behavior will contain details that would make it difficult to follow or assess, which is why a selection is necessary. In order to create a transcript with an appropriate level of detail a number of choices need to be articulated. The transcriptions were to be the main source of data and the transcription was focused on creating a text that would be easy to read and analyse. For this reason, verbal language markers such as shortenings of words and vernacular use of different tenses were replaced with more or less grammatically correct versions. This 'washing' of the language was done with consideration taken to the level of detail in the following analysis. Non-verbal behavior such as laughter, giggling and sarcasm was noted

when it was considered to significantly alter the interpretation of the utterances (Gee, 2011; Ochs, 1979). The video material was recorded to indicate which mathematical text the teachers were discussing at a given moment in case the transcripts failed to reveal this, but in the analysis this never happened and, consequently, the video material was never used.

Analysis

The transcriptions of the interviews were analyzed in four steps. The first two steps involved an effort to find and analyse frequent words or phrases in connection with the part of teachers' discussions that are centered on interpreting, understanding and assessing the mathematical texts. The first search, in which the words *think*[1], *understand*[2] *and see*[3], along with their different tenses, were identified, was manual. In the second search, a computer was used to identify and indicate instances in which the above mentioned words appeared. In the third step, the indicated instances in the transcription were analyzed through questions regarding who the agent was, that is who was doing the thinking, understanding and seeing. These were followed by questions regarding what it is that one thinks, understands and sees. The fourth and final step of the analysis was qualitative and the questions posed were: what practices are being enacted in the discussions? What is being made significant and how is this accomplished? What are the differences and similarities in the different ways these words are discussed?

Discourse analytic approaches to research are concerned with the way language is used in a situation, and the underlying assumption is that the study of how language is used in a particular context can give insights into the practices and activities that are relevant for the participants (Morgan, 2014). Having identified particular instances of action along with agents and objects in the first three steps, the analysis was then directed towards investigating the practices and significance indicated by those instances. The analytic process included an effort to identify possible differences and

[1] The Swedish word **tänk** will generate computer search hits on all the different tenses for the word *think*.

[2] The Swedish word **förstå** will generate computer search hits on all the different tenses for the word *understand*.

[3] The Swedish words **se**, **såg** and **sett** which correspond to *see, saw and seen* were used in the search.

similarities in the instances in order to create categories. Even if the questions that are posed to the text are presented, and even if these questions are grounded in discourse analytic ideas that have been presented in earlier research (see Gee, 2011, p. 211), such a process is still more difficult to account for than any quantitative process. It is dependent on interpretations and ideas that are unique to the individual researcher. It is possible to claim that a word appears a certain number of times but more difficult to claim that this represents an indication of a certain phenomenon, or in my case, practice. For the process to render valid and reliable results I have to be prepared to repeatedly revise my categorization in the face of critical questions, from myself and from others, and the process has to be subjected to external review. The categorization was indeed revised and the study was subjected to external review in different contexts several times. One example of this revision is that one of the initial research questions had been aimed at identifying teachers' *approaches* to interpreting, understanding and assessing mathematical texts. An approach could be seen as associated with an intentional act. A teacher can change her approach toward a phenomenon, and this change is then the result of some form of reflection and decision. The group discussions did not provide support for the existence of such intentions. Instead, it seemed that the teachers worked in different modes with which certain perspectives could be associated. The research question was therefore rephrased and the results revised.

Study design for the third study

The third study investigates young students' mathematical writing by examining mathematical texts created in response to a mathematical problem-solving task. The aim of the study is to disassemble the students' writing by examining the different resources that students use to document and communicate their problem solving. The study draws on multimodality, which is not a theory but rather a term that maps a domain of inquiry (Kress, 2011). As described in the theory section, multimodal approaches to research are built on theories on communication that originate in social semiotics (Kress, 2010b) and Halliday's Systemic Functional Grammar (Jewitt, 2011a). The most important theoretical assumption that underpins multimodal approaches is that, in communication, meaning is made through a range of modes, where language is just one. Language is seen as part of an ensemble of resources for meaning making that also include image, gestures, gaze, music and posture, to name a few (Jewitt, 2011b). Another important starting point is the recognition that meaning making occurs in social contexts and that in this context each mode is doing communicational work (Hodge & Kress, 1988; Jewitt, 2011b). The starting point of the study is that even young students orchestrate meaning through their choice and organization of different modes in their written communication.

The study's interest lies in investigating how students write, what communicational choices they make and what semiotic resources they use to realize their communicational intentions. Such an interest assumes that writing can be separated, at least to a certain extent, from the content. A student who is documenting her problem-solving process may, for example, choose between different ways of highlighting an answer or between using words or images to account for the process. As demonstrated in the second study, such choices, no matter how trivial they may seem, influence teachers' assessments of students' texts. At the same time, these choices may have more to do with the student's conception of an ideal documentation of problem solving than with her ability to accurately and effectively solve the problem. In mathematics education many researchers who have worked with students' written representations have taken an interest in the various choices that students make from a mathematical point of view (see for example Duval, 2006; Goldin & Shteingold, 2001; L. Radford, 2014; Smith, 2003). In this study I am interested in students' choices from a communicational point of view. When given a choice of

mode, what motivates a student to choose an image over words or numbers when delivering an answer to a problem? Given that my interest has more to do with the communicational than the mathematical aspect of mathematical communication I have come to rely on some general research in multimodality, originating in Halliday's research, rather than research in mathematics education. I argue that the mathematics education community can benefit from this perspective.

As noted above, social semiotic approaches acknowledge the importance of social context in the analysis of text. What, then, is social context, and how much information about context is necessary for a researcher to be able to say anything about a text? Two extreme positions can be identified in a debate about context in relation to text analysis. Some conversation analysts may argue that anything outside the text is irrelevant for its analysis, whereas some ethnographers may argue that as much of the context as possible must be taken into account. The position taken by social semioticians and discourse analysts tends to be located somewhere in the middle. The study adopts a social semiotic approach where context needs to be taken into account but where text analysis does not require ethnographic accounts. Morgan (2006) has argued that there are two different types of context that need to be taken into account when analyzing student texts: the immediate situation in which the texts were created and the broader culture within which the students are embedded.

For this study, the immediate situations in which the empirical material was created were designed by ten teachers from eight different Swedish schools, all of whom had agreed to collect and forward accounts of mathematical texts from a problem-solving exercise with their students. The students who produced the texts were 9 to 12 years old, and from them a total of 519 texts were collected. Information gathered to account for as much of the context as possible included information about the schools and about the teachers, teachers' instructions, formulations of the problems and information about infrastructural resources like pens and paper. The schools are from two municipalities, one midsized and one small, and all schools but one are run by the municipality. All the schools have students from a varied socioeconomic background. As important as it is to know things about the background it is also worth considering that in connection to national tests in mathematics there are practices in which contextual information is sometimes thought of as compromising. As with

texts in any other subject, the texts that students produce in mathematics are meant to be understood outside of the context in which they were created.

Another part of the immediate context is the mathematical problem. Documenting mathematical problem solving in a school context typically requires reporting not only the correct answer but also accounting for the problem-solving process. Writing in connection with problem solving thus requires a student to produce a rather complex text. Different problems invite students to adopt different strategies and subsequently lead to different types of documentation. The choice of problem is therefore important. The problems chosen for this study are of a particular type. They are Diophantine equations involving the identification of a number of ways to distribute, for example, legs on animals such as hens and pigs or wheels on vehicles such as motorcycles and cars. The problem can be formulated in such a way as to limit the number of possible combinations or as an open problem to which there are an endless number of solutions. A small number of legs or wheels also results in a small number of combinations; the problem can therefore be adapted to fit different students or age groups. The students can also be asked to demonstrate that they have found all possible combinations and explain how they know this. The problem offers opportunities to adopt a more or less systematic trial-and-error strategy, but there are also other ways to solve the problem. Given that the problem involves concrete objects it also offers students opportunities to draw. All these properties contributed to the choice of the problem type.

Communication, both written and oral, in combination with problem solving, is part of many mathematics syllabuses worldwide (NCM, 2014). Written and oral communication is also a part of the PISA mathematics framework (OECD, 2013). The broader cultural context for this data collection thus indicates that writing and problem solving are important parts of mathematics education internationally. The study takes place in Sweden, whose school curriculum lists communication as one of five abilities that students should develop. In the knowledge requirements in the Swedish mathematics syllabus, pupils are also expected to be able to describe and discuss their approaches and to use diagrams, symbols, tables, graphs and other mathematical forms of expression with some adaptation to the context (Knowlede requirement for the end of school year 6, SNAE,

2011, p. 65). Mathematics teachers in Sweden have also been encouraged, through a national professional development project, to engage students in problem-solving activities and encourage them to communicate. This project represents an attempt to strengthen the quality of mathematics teaching nationwide and ultimately improve student learning. The project was introduced by the Swedish National Agency for Education in 2013 and aims to reach close to 40,000 mathematics teachers. It is the most extensive professional development endeavor ever in Sweden.

A text could be said to constitute a mathematical text only if it is recognized as such by the community in which is produced and interpreted. In this study the term 'mathematical text' refers to the written work produced by a student in response to a mathematical problem-solving task. The definition of *text* is extended to include non-language elements such as images, numerals, graphs, tables, symbols and other resources for meaning making. A text is dependent on media although there are a number of different media for texts. Although paper as the primary medium in education has, to an increasing extent, come to be replaced by screens, the schools that were part of the study still rely heavily on paper; hence, all 519 texts were paper texts. All students also had access to colored markers, and in most student groups they were also able to choose between graph, ruled and plain white paper. The texts were coded as they were collected from each teacher to ensure that they later could be identified as originating from a particular student group. This also provided an opportunity to compare not only different student groups, but also different age groups.

Analysis

As stated above, the aim of the study is to deconstruct the students' writing by examining the different resources that they use to document and communicate their problem solving. The study draws on multimodal discourse analysis. This approach is focused on examining the semiotic potentials and resources that are available to text makers in a particular context and aims to describe the choices which are taken (Jewitt, 2011a). In this study this is done by investigating the modes through which students communicated as well as the uses to which these modes are put.

In the third study the identification and categorization of the different modes of communication that are used in the 519 texts forms a base from

which the remaining analysis departs. Modes of communication can be categorized in different ways and be based on various definitions of what constitutes a mode (Kress, 2011). For the purpose of analysis the contents of the texts are categorized as belonging to five different modes: image, words, numerals, mathematical symbols and layout. Drawings, including tallies, are thought to belong to the mode of image, but lines, arrows, space and distance were considered resources for layout. Abbreviations, even idiosyncratic single-letter examples, along with units are thought to belong to the category of words. All numerals, whether they are numbers or nominals, are included in the numerals category. In the category of symbols only mathematical symbols are included, which leaves out punctuation marks from the analysis. This first categorization acts as a premise for the second analysis. There are other possible categories of modes. The decision to use these five is the result of a long process. These modes are not previously established as modes in the formal sense described by Kress (2011) and accounted for in the theory section above. Nevertheless they do cover and describe what the students appeared to be doing in their texts, which can be seen as an indication of their usefulness as analytic tools.

The second part of the analysis involves identifying what the different modes are used for through a conventional content analysis as described by Hsieh and Shannon (2005). This type of analysis is most common in studies whose aim is to describe a phenomenon and where existing theory or research literature is limited and there are no preconceived categories. When categories are constructed from the data, the method of analysis can also be described as inductive category development (Mayring, 2000). A random sample of 15 texts was analyzed by focusing on one mode at a time, and the various uses identified forms the base for the analysis of the 519 texts. The initial uses that were identified were condensed and complemented during the process. One example of a category of uses that was added to the initial categories was that of using words as well as numbers to denote or explain drawings or calculations. In the analysis the uses are identified through their role in the text as a product for the purpose of communication. This does not exclude multiple uses of modal elements, nor does it exclude personal or cognitive uses for different parts of the texts. The analysis does not claim to identify intentions, but rather apparent purposes. Information on what the students are actually thinking with

every communicational choice is difficult to access, even for teachers who know the students.

When an apparent use of a certain mode was identified in the analysis of the texts, it was noted on a spreadsheet as either employed (1) or not (0) on all texts. This way of coding offers opportunities to compare student groups and age groups, and it also provides quantitative information of how common a certain practice is across the sample as a whole.

Study design for the fourth study

The fourth study builds on the third in that it employs the same type of analysis, although the data collected for the fourth study includes digital material as opposed to the pen-and-paper texts that constitute the data for the third study. In this study the immediate situation in which the data was collected was designed in collaboration with one teacher. This teacher was involved with the planning and execution of three problem-solving activities in her classroom, from which analogue and digital texts, produced with the help of an interactive whiteboard, were collected. The teacher and I discussed the various methodological choices for these situations in relation to what was thought to benefit the students as well as the study.

Added to the data of the fourth study were also short interviews with the students after solving the third problem by using the interactive whiteboard. The interviews were focused on how the students perceived the difference between analogue and digital writing. The interviews were partly transcribed and analyzed separately from the mathematical texts.

The teacher who was involved in the study design and the data collection had participated in two earlier studies and shown an interest in a continued participation in research. This teacher had also worked with digital tools such as an interactive whiteboard and tablet computers for several years. She was open to any type of problem-solving activity and interested in using the results of the study to discuss written communication with her students.

The 14 students were sixth graders, 12 years old and, at the time of the last problem solving, they had just started their last term at their elementary school. The following term they would all begin lower secondary

school. In the first problem-solving activity 12 students participated; two were absent that day. During the first problem-solving activity, which was organized by the teacher, the students created their texts using pen and paper. In the second problem-solving activity, also organized by the teacher, all 14 students participated in solving a second similar problem. This time they were instructed to solve the problem using pen and paper and then to record their problem solving using the interactive whiteboard. This dual recording of the same problem-solving process was the result of a misunderstanding between the teacher and me. The second problem-solving activity was not supposed to involve pen and paper at all. The misunderstanding resulted in a second set of paper texts and a new set of 14 digitally produced texts. The 28 texts which were the result of this second problem-solving activity and the 12 texts from the first problem-solving activity were analyzed with the aim of selecting 8 students to interview in connection with the third problem-solving activity, which was supposed to be entirely digital. The selection process aimed to provide diversity in relation to use of strategy, use of resources, overall layout and gender. Four girls and four boys were selected. The third problem-solving activity was organized by me. One of the selected boys was absent that day, and when the process of using the interactive whiteboard to solve a problem and interviewing the students proved to last longer than expected I decided to include only 7 students in the data. These students had produced 4 texts each, one paper text in which the first problem-solving activity was recorded, one paper text and one digital text in which the second problem-solving activity was recorded and one digital text, which was the recording of the solutions to the third problem. These 28 texts were analyzed in the second stage.

The organization of the third problem-solving activity, in which the students were supposed to document their problem-solving process using the interactive whiteboard alone, is artificial in the sense that this is not something students normally do. They use the IWB often, but it is used in whole-class situations where students have the opportunity to complement their writing with verbal explanations. Creating a situation in which students were alone is thought of as a way to investigate the communicational choices they make in relation to the communicability of their texts, now that they lack the opportunity to complement their texts verbally.

The interviews were conducted in connection with each student's problem-solving process. The students were alone with the whiteboard and when they felt they were ready we looked at the board together and I asked them questions regarding what they had written and why. The students were asked to compare their digital writing and the writing they had produced earlier with pen and paper. I provided the earlier work for the students who wanted to see it for comparison. The final question of the interview dealt with the perceived reader of the students' texts. Who are they writing for and what do they feel they have to explain to this imaginary reader? This question is thought of as a way to compare their ideas about the perceived reader and their texts.

The problems that the students solved were all variations of the same Diophantine equation that had been used in the second and the third study. As a way to vary the problems the context was changed to packing computers into two different sixed boxes, filling two different sized jars with jam and making two different sized marzipan figures for Christmas. The first two problem-solving activities were two months apart and the third problem-solving activity was organized six weeks after the second. The means to create texts varies in today's schools, as students use pens or pencils on paper or on interactive boards. Most interactive boards also allow students to use their fingers to write and the texts in this study were created with pens or fingers on paper or interactive boards.

Analysis

The fourth study employs an analysis that is focused on examining the resources that are used by students, but, unlike the third study, these resources are analyzed in relation to a preconceived set of text elements which are taken from PISAs description of expressive communicational ability. These includes 1) showing the work involved in reaching a solution and/or summarizing and presenting intermediate mathematical results, 2) constructing and communicating explanations and arguments in the context of the problem and 3) articulating a solution. In the initial stages of the analysis, the five modes from the third study were thought of as a natural starting point in the identification of resources. These modes, however, proved insufficient in the analysis of the digital texts where students used, for example, templates for geometric shapes, features for creating lines and several different colors. The concept of *communicational resource*, as described by Blommaert (2013), was used instead of *mode* in

order to facilitate the analysis by providing less restriction in the face of the significant differences between digital and analogue writing. The resources which Blommaert describe are infrastructural, graphic, linguistic, semantic, pragmatic, meta-pragmatic, and social and cultural (see article 4). These resources, along with the preconceived categories for communicational ability, offer a new dimension of analysis, compared with the third study, which contributes to the overall result of the thesis. The differences between the analytic tools used in the third and fourth study offer an opportunity to contrast the two studies and reach a deeper understanding of the various dimensions of students writing in school mathematics.

Only the parts of the interviews in which students talked about the differences between analogue and digital writing and where they talked about for whom they were writing were transcribed. Unlike the analysis of the texts, the analysis of the interviews focused less on the students' use of resources and more on their thoughts on writing.

Method discussion

Research requires decisions regarding methods for collecting or creating data, for handling data and for analyzing data. All such decisions have consequences for the results of the research, and a researcher has to be aware of and ready to defend these decisions. In order to defend them one first has to look at them critically to identify their strengths and weaknesses. Below I discuss the decisions for each study which I regard as problematic in relation to questions regarding the validity, reliability and generalizability of the results of the study.

In the first and second studies three international competency frameworks for mathematics were analyzed and 19 mathematics teachers were interviewed. As with any qualitative research it is possible that one finds what one is looking for. Previous research pointing to the same conclusion, that communication is relatively unproblematized, may have led to bias in the analysis. Such bias may also lead to a situation in which data in the analysis that points in another direction is unconsciously omitted because it did not fit with a preconceived idea. Paradoxically, the fact that earlier studies are confirmed by the result of my studies may also be seen as strength, in the sense that they did not find something completely different, but rather added new data to a well-known problem. That the results confirm the findings of previous studies adds to the validity of these studies, but only if

they are conducted in a reliable way. Weaknesses that can be identified in the first study are mainly connected to the analysis. As described earlier, questions about the validity and reliability of research that adopts discourse analytical approaches are associated with several different aspects. Gee (2011) pointed to the elements convergence, agreement, coverage and linguistic detail as especially important for assessing the validity and reliability of discourse analysis. The first study includes a rather thorough linguistic analysis and is partly modelled on the analysis of an earlier study. The linguistic detail of the analysis and the transparency of the model of analysis contribute to convergence, as there are several aspects that point in the same direction. The carefully researched analytic questions along with the definition of communication also contributes to the validity of the analysis as it provides coverage across several important aspects of communication and also adds to the transparency. As with most research the analysis was also scrutinized through academic peer-review in several stages of the analytic process, which also adds to the validity of the analysis through agreement.

In the second study the aspects that may be open for critique are, to a larger extent, connected to the collection of data. The teachers were selected by their school principals and asked to participate in the research. Whether these teachers represent average teachers or a particular group with ideas that might stand out in the teacher collective is difficult to know. There is so far nothing in the interviews, transcription or presentation of the results to different groups of teachers that implies that the teachers involved represent views that are uncommon. The teachers for whom the results have been presented express a genuine interest that seems based in recognition of the problems associated with assessment of students' texts. A second weakness in the second study is the organization of the interviews. As presented earlier the group interview was seen as a way to access the ideas that are part of discussions in the teacher collective rather than the teachers' individual ideas. How can one trust a group discussion to uncover such ideas? What if the situation of being involved in a research interview constrained the teachers, leading them to say and do things they normally would not? Given that I acted as an interviewer in the sense that I did ask questions, there is cause to categorize these meetings as interviews (Holstein & Gubrium, 2011). These interviews could, however, be characterized as group discussions in the sense that, during the course of all five interviews, the interviewees themselves often acted as

drivers of the discussion. This could be seen as evidence of interest as well as an indication that the teachers trusted me as an interviewer to follow the discussion (Miller & Glassner, 2011). The interviewees agreed with one another on some occasions but disagreed on others, posed their own questions to the texts and introduced new topics as well as questioning the topics that were given. They were often interested in, and even fascinated by, their colleagues' arguments. The teacher discussions thus included both 'exploratory' and 'expository' talk (for details see Crespo, 2006). Such a pattern of interaction provided an opportunity to extract the meanings that were taken as shared as well as the meanings that were contested and/or negotiated by the teachers. Hence the pattern of interaction, as it developed, may be seen as contributing to the validity of the data collection.

In all the studies the analysis of the transcribed interviews employed a discourse-analytic approach. The validity of this approach is dependent on the reliability of the interpretation (Jaipal-Jamani, 2014). It can be argued that familiarity with the particular language game that is associated with mathematics teachers' discussions adds to the reliability. It can also be argued, however, that too much familiarity constrains the possibility of approaching the transcripts from the interview systematically while paying attention to detail. My familiarity with the language game of assessing students' mathematical texts is fairly extensive. I taught mathematics to 13-16 year-olds for 15 years and I spent 8 years teaching in a teacher training program. What I am not familiar with, however, are the particular ways in which mathematics teachers who teach 9-12 year-olds discuss and assess these texts. It can therefore be argued that I have the familiarity needed to infer meaning from the teachers' discussions, but I also have, at least partly, an outside perspective (Miller & Glassner, 2011).

If validity rests on the possibility of confirming the results of earlier studies, then a possible problem with the last two studies is that they have few precursors. They differ from other studies in two respects; they investigate students' writing in mathematics with a focus on their communicational choices rather than their mathematical strategies, and they are focused on young students, aged 9-12. There are studies of secondary or tertiary students' use of representations (see for example Duval, 2006; O'Halloran, 1998) and on the way professional mathematicians write (Burton & Morgan, 2000), but as far as I know there are no studies of young stu-

dents' mathematical communication that has investigated the communicational aspects rather than the mathematical. Given that there were no models on which to build the analysis; the last two studies relied on theories and concepts from other fields. Since the studies focused on the communicational rather than the mathematical choices, and particularly the integration of mathematical language and natural language, it was natural to make use of concepts and models from social semiotics, multimodality and sociolinguistics.

In relation to the generalizability, a possible weakness in the third study is the limited data. I would argue that the number of texts (519) is sufficient to a certain extent for such an analysis, but it may be seen as problematic that they all deal with the same problem type. To present a comprehensive inventory of the communicational resources available to young students in their documentation of mathematical problem solving the sample of texts would have to include several different problem types. The inclusion of other problems would likely yield a different list of uses to which the different modes were put. The focus on one single problem type, however, did offer an opportunity to present a relatively comprehensive inventory of the communicational practices applied to this particular mathematical problem. Whether different problems to solve would lead the students to make use of other modes is difficult to know. What counts as a mode, or resource as in the fourth study, also depends on the context.

The validity of the results of the fourth study is connected to its design. The fourth study was a small-scale investigation that involves the use of an interactive whiteboard. Even if interactive whiteboards constitute a relatively new phenomenon in mathematics education in Sweden, the students in the study demonstrated that they were experienced users. The IWB was used in mathematics to share solutions to problems with the rest of the class in real time, but also to record and save such solutions. To use the IWB to solve a problem while at the same time documenting the problem-solving process privately rather than in interaction with peers was, thus, a new situation for the students. They had considerable experience when it came to privately documenting mathematical problem solving using pen and paper as well as experience using the IWB to show their solutions, but the combination of these activities created an artificial situation that was new to all the students. Given that the interest of the analysis was that of investigating the communicational aspects of students' mathematical texts,

both analogue and digital, I argue that the creation of an artificial situation through the combination of two well-known situations offered opportunities to examine the communicational choices and the resources that students drew on. In this way the artificial situation can be seen to add to the validity of the study.

In comparison to the other studies in the thesis, the last study is relatively small. It investigates a small case of seven students in a special situation. The results can be seen to confirm and complement the third study, which also investigated students' use of different resources in their creation and design of mathematical texts. Given the small scale, it may be difficult to generalize from the results of the fourth study. On the other hand, it can be argued that results of the last two studies converge in a way that adds to their validity and generalizability. The different practices of the specific students are not generalizable, but the demonstrated variety in use of resources and the adaptation to different situations can be seen as representing the sophistication of students' communication.

Ethical considerations

Ethical consideration is a critically important aspect of ensuring that research processes and findings are trustworthy and valid. The term ethics derives from the Greek word ethos, meaning character (Corts, 1968) and it could be argued that the outcome of any research depends to a large extent on the 'character' of the individual researcher. Researchers, however they are funded, could be said to have obligations to a number of groups. They have obligations to society, funders or employers, colleagues and to subjects (SRA, 2003). Obligations to society include a responsibility to maintain high scientific standards in the methods, collection and analysis of data, and the presentation and assessment of findings. Obligations to funders and employer include upholding professional integrity as does obligations to colleagues. Social research is dependent on maintaining standards and appropriate professional behavior in the professional research community which requires methods, procedures and findings to be open to collegial review. Obligations to subjects include striving to protect subjects from harm as a consequence of their participation in research. Subjects' participation should be based on their informed consent, the confidentiality of data and openness on the use of results.

The aim of this thesis has been to add to the body of knowledge on young students' writing in school mathematics. As a publicly funded researcher I have strived to contribute to the research community and to the teaching practice of mathematics teachers, by addressing what I feel is an area of concern for mathematics education and for teachers of mathematics in particular. I would argue that research into young students' writing in school mathematics has been limited, although writing is an important part of mathematical activity. Investigating unarticulated ideas on communication, held by teachers, and expressed in authoritative texts, also contributes to a deeper understanding of the conditions for students' writing. In addressing what I consider to be important issues in school mathematics I feel I have fulfilled part of my obligation to society. By reporting my research into these issues in the form of scientific articles that have been subjected to peer-review on all stages of their creation, I feel that I have also fulfilled part of my obligations to my funders, the Swedish state and my hometown municipality. In relation to those who have funded my research, as well as in relation to my colleagues, I have also upheld my professional integrity by subjecting myself and my work to a close examination in different contexts, academic and professional, such as seminars, conferences, working groups, courses, lectures and public examinations. In one respect I have also upheld my professional integrity by taking an obvious ideological stand, inspired by Anna Sfard. Throughout my text(s), in every instance where an unidentified person is the subject, I consistently use the pronouns 'she' and 'her'. The reasons for this are stylistic as well as ideological. Firstly, the English language lacks a neutral pronoun, which sometimes creates situations in which the convention is to use the somewhat awkward construction 'he or she'. Secondly, in a world where, even linguistically, male is the norm, I feel I have a responsibility to contribute to an academic language in which female pronouns are equally viable.

The obligations that researchers in social research have to their subjects are of critical importance (SRA, 2003; SRC, 2011). In the second study the subjects were teachers from various schools in two different municipalities. They were informed about the study and asked to participate through a letter. In connection with the group interviews, the teachers were again informed of the purpose of the study and each of the teachers was also reminded that regardless of what they had previously agreed to, in relation to their school principal or in relation to me as a researcher, they were under no obligation to stay for the duration of interview should

they for any reason lose interest or feel uncomfortable. Once the study was completed and reported in the form of an article accepted for publication, the groups of teachers were invited to follow-up meetings in which the results were presented and discussed. Three of the five groups accepted the invitation, and in these meetings, the teachers all seemed to find it interesting to hear about and have the opportunity to discuss the results.

In the third study, students' texts were collected from ten teachers. These teachers had been informed of the purpose of the study and asked to conduct problem-solving activities that differed as little as possible from those that were a normal part of their classroom practices. The teachers informed their students and the parents of the students about the study in ways they deemed appropriate. The teachers had been instructed to ask their students to not include their names in the texts, but there are several instances where students did anyway. Prior to the analysis the collected mathematical texts were coded, and in the analysis, each group and each text were assigned a code. This was seen as a way to ensure confidentiality.

The fourth study took place in one class with one teacher who had shown interest in sharing her work on several earlier occasions. This teacher participated in the planning of the study and she carried out the activities from which she collected the data in ways that she felt were comfortable for her and her students. She distributed a letter containing information on the purpose of the study as well as its design to the parents of her students, and she also discussed this information with her students. The parents and the students were informed that although some students would be interviewed in connection to their problem-solving activities, the interest was not in their general mathematical competence, but rather the way they communicated their problem solving in writing. This was even talked about among the students and one student started the interview by saying "I know that you don't care if it's right or wrong". This student claimed to have considered this reassuring and also said it made him more comfortable. I took this as an indication that the students were not anxious about being assessed based on their mathematical performance. The information to the students and their parents also stated that data from the problem-solving activities and the interviews would be recorded with an audio recorder and that notes would be taken. The letter also stated that the identity of the students as well as the school would be protected and

that data would be handled in such a way as to guarantee confidentiality as much as possible.

In each of the studies in the thesis there have been ethical considerations at nearly every stage of the process. As the origin of the term *ethics* suggests, it has to do with character. Research processes demand that a researcher question her motives, her assumptions, her methods, her results and her conduct with a rigor that is seldom a part of her regular practice. Uncertainty about choices is present at every stage. The only way to move forward with a process of this kind is to subject oneself, one's choices and their consequences to scrutiny, be prepared to listen to the opinions of others, and to revise one's work again and again. This requires a particular kind of character, one I hope I have cultivated over the course of this research process.

Results

In this section, the results of the thesis are presented. The three research questions are presented in relation to the studies that set out to answer them. The relationship between the four studies will be further presented in the discussion following the results section.

The purpose of the thesis is to examine and problematize students' writing in school mathematics and the various understandings of the relationship between students' written communication and their achievement. Given that writing is both a means and a goal in school mathematics, and given that students' writing is used to infer ideas regarding a student's level of knowledge or skill, writing stands out as an important area for research. The need to know more about students' writing acted as the motive for the thesis. Four studies were conducted with the aim of reaching a deeper understanding of writing in school mathematics, through the examination of how the concept of communication is described in authoritative texts, how students' writing is viewed by teachers and how writing is employed by students. These different aspects of writing are expressed in the three research questions below:

A. What communicational logic is embedded in international authoritative texts in mathematics education, and what are the possible consequences for teaching and learning?
B. How do teachers interpret, understand and assess students' mathematical writing?
C. How do students use different communicational resources in their mathematical writing?

Communication in school mathematics is a means and a goal, and it is both taught and learnt in the mathematics classroom. A starting point for the thesis is that there are different conceptions of and logics about communication which affect the teaching, learning and assessment of mathematics and which are, therefore, important to investigate. By exploring and problematizing the concept of communication in mathematics education, this thesis can add to a deeper understanding of how communication is viewed and how this affects the teaching and learning of mathematics. The concept of communication was explored and problematized through

the investigation of three internationally renowned competency frameworks in mathematics. Authoritative texts are texts that are thought to influence mathematics education and the first study examined examples of such texts, three competence frameworks for mathematics, with the aim of identifying their communicational logic. Such logic is part of the broader cultural context in which writing takes place, and the frameworks for mathematical literacy that were investigated constitute an important part of this broader cultural context.

The first study aims to answer the first research question: *what communicational logic is embedded in international authoritative texts in mathematics education, and what are the possible consequences for teaching and learning?* The findings suggested that these frameworks operate with a rule that states that mathematical communication, even at school level, is independent of addressee. This view stands in opposition to theories that view communication as dependent on the social context in which it takes place. The view is, however, consistent with a cybernetic tradition in communication in which the focus can be directed entirely towards the sender and her message. Findings also suggest that mathematical communication involves language which is preconceived and ready-made. The frameworks offer different descriptions of this language, wherein its features are either related to form, for example *numeric, formal* and *technical*, or to objective qualitative properties, such as *clear, precise* and *concise*. Such a view of mathematical language connects meaning to the language itself rather than to the social context in which it is used.

Taken together, these rules express a logic that casts mathematical communication as being both transparent and unproblematic. A possible consequence of this logic is that communication risks becoming invisible to teachers and students, given that communication appears to pose no problems to those who know the content. Such a view assumes that if a student knows the mathematics involved she also knows how to present this adequately. The question of whether it is possible to tell the form from the meaning and, by extension, to differentiate the ability to communicate in mathematics from the ability to successfully exercise mathematical activity is not interesting, as there is no difference between the two. There is a risk that students who might be struggling with the documentation of mathematical activity rather than the actual activity will be misunderstood and,

consequently, unable to develop good communicational skills. In order to develop students' ability to communicate adequately in writing the problems regarding this writing need to be highlighted not made invisible. Teachers' ideas about students' mathematical writing was investigated in the second study.

The second study addressed the second research question: *how do teachers interpret, understand and assess students' mathematical writing?* The study involved group discussions/group interviews with teachers regarding the way they interpreted mathematical texts produced by young students in response to problem solving. The results indicate that two different modes are visible when the teachers discuss the mathematical texts. The first is a pedagogical mode that is connected to the teachers' roles as teachers or pedagogues and where identification and understanding of student's strategies is foregrounded. The second is an assessment mode which is connected to teachers' roles as examiners. This mode adopts a deficiency perspective where the students' texts are measured against an implicit and unarticulated ideal mathematical text and where particular features are found to be 'missing'.

Pedagogical mode

- Connected to teachers' roles as teachers or pedagogues
- Focused on understanding
- Sympathetic and lenient
- Several ideas and strategies for helping students develop better problem-solving strategies

Assessment mode

- Connected to teachers' roles as examiners
- Focused on students' communicational choices
- A deficiency perspective – what is 'missing'
- Reference to a vague and unarticulated 'ideal' text
- Very few ideas on how to help students write better mathematical texts

The two modes, the pedagogical and the assessment mode, deal with different aspects of the mathematical work. In the pedagogical mode, the teachers' discussions are focused on understanding what the students have done and the teachers are focused on using this information to inform

future teaching. In the assessment mode, however, the teachers focus almost exclusively on the communicational aspects of the mathematical texts, and the discussions rarely go beyond stating a certain level of achievement. In the assessment mode the teachers seem aware that writing is an ability that can be developed through teaching, but there are few examples or suggestions as to how this can be done, and it seems that this is not a part of teachers' everyday practice. This constitutes one important difference between the two modes. When the teachers turn to the pedagogical mode, their perspective is broad and they discuss not only their interpretations but also the possible consequences of these interpretations. In the assessment mode the discussions are focused on what is missing and occasionally on why it is missing from a student's text; different ways of helping students identify and subsequently develop "good" communication strategies are not part of the discussions. In the assessment mode it also becomes evident that teachers value the different modes that students use differently. The mode of image is seen as a transitory cognitive tool and it is not acknowledged as adequate for communicating either the problem-solving process or the answer to the posed problem. Students' use of different modes, such as image, in their documentation of mathematical problem solving was investigated in the third study.

The third study addressed the third research question: *how do students use different communicational resources in their mathematical writing?* The study investigated a sample of 519 mathematical texts by students aged 9-12. The findings of the study indicate that students have access to and make use of a number of communicational resources as they attend to different questions regarding their design of their mathematical texts. The communicational choices connected to documenting a problem-solving activity are numerous, as is illustrated by the fact that students demonstrate many different ways of organizing their texts and integrating mathematical and everyday language. The great diversity indicates that students have very different ideas regarding how, what, for whom and why they are writing. Diversity can be found even in texts from a single group of students from the same class who are assumed to have received similar instructions and prompts and to share certain norms for communication. It can be argued that the older the students are, the more they work towards facilitating communication with a reader who is unfamiliar with the context. They do this by adding elements to their texts that serve to explain the context and various other components such as units and draw-

ings. There are, however, several examples of texts from younger students that exhibit the same kind of reader focus, and it is difficult to say that age predicts such communicational choices. The third study resulted in an inventory of various elements in students' mathematical texts along with the different modes used. The inventory can be seen as evidence of the diversity found in the students' texts, and although it is not exhaustive, it has the potential to assist teachers in their discussions with students regarding what is considered appropriate writing. The inventory can act as a tool for developing students' writing by introducing questions such as what, how, for whom and why students should communicate in mathematics.

Table 2 *An inventory of various elements in students' mathematical texts along with the different modes used.*

	IMAGES	**WORDS**	**NUMERALS**	**SYMBOLS**	**LAYOUT**
Title of the text		X			
Stating conditions for the problem	X	X	X	X	
Accounting for the problem-solving process	X	X	X	X	
Illustration	X				
Explaining or denoting units		X	X		
Meta-text, structuring of text, explicating	X	X	X		X
Stating an answer to the problem	X	X	X	X	

The third study investigated students' traditional writing when they are using pen and paper to document their problem solving. The fourth study, which also aimed to answer the third research question—*how do students use different communicational resources in their mathematical writing?* — focused on students' digital writing. This study investigated and compared students' use of various resources in their design of analogue and digital texts in relation to mathematical problem solving. Like the third study the findings of the fourth study indicate that students have access to a number

of different resources when writing and designing mathematical texts. The students draw on infrastructural, graphic, semantic, linguistic and social resources to structure and create meaning in their texts. Even if the differences between the digital and the analogue texts are small the findings indicate that, when viewed as products for communication, the digital texts contain less elements such as transition markers, explanations and structuring devices that serve to facilitate the reading of the text. As a result of this the texts also display less internal coherence than their analogue counterparts. Another difference is the use of color and the approach to linearity.

The students all created their texts with a reader in mind – ranging from their teacher or the researcher to a description of an unknown person for whom they would have to explain a number of things. Given that the use of the IWB is associated with a situation in which verbal explanations can complement students' texts, the situatedness of the writing and design of the digital texts makes it difficult for the students to free themselves from the context, and to provide an unknown reader with details on, for example, the context. This can be interpreted as an indication that students do take such aspects into account when they design their digital texts.

The third and fourth studies together indicate that young students have the ability to attend to questions such as how, what, for whom and why when they design their texts. The students seem to manage the merging of the two functions of writing for oneself and writing for others well, and many of them also seem to consider different readers for their texts. They employ a number of resources to explain not only their general problem-solving strategies, but also the different elements that go into their text. The analysis of students' writing in the third and fourth studies contributes to a theoretical understanding of the communicational choices involved, which offers teachers opportunities for a more informed approach to teaching in order to develop students' communicational abilities.

The purpose of the thesis was to examine and problematize students' writing in school mathematics and the various understandings of the relationship between students' written communication and their achievement. The results of the four studies indicate that students' writing is more complex than is acknowledged by teachers and authoritative texts in mathematics education. Results point to a sophistication in students' approach to the

merging of the two functions of writing, writing for oneself and writing for others. Results also suggest that students attend, to various extents, to questions regarding how, what and for whom they are writing in school mathematics. The relationship between writing and achievement is dependent on students' ability to have their writing reflect their knowledge and on teachers' thorough knowledge of the different features of writing and their awareness of its complexity. The results of the thesis contribute to a deeper knowledge on students' written communication in school mathematics.

Discussion

The purpose of this thesis was to examine and problematize students' writing in school mathematics and the various understandings of the relationship between students' written communication and their achievement. The contribution of the thesis to the field of education in general, and the field of mathematics education in particular, lies in the problematisation of the concept of communication for the purpose of assessment and, in particular, the relationship between the knowledge that writing reflects and the knowledge required for writing adequately. Given the crucial importance of communication [writing] in teaching, learning and assessment in education, there should be room for more studies in this field; micro studies on the particular writing on a specific task as well as macro studies on what constitutes adequate writing. The communicational perspective on mathematical writing upon which this thesis is based contributes to a deeper knowledge of the different aspects of students' writing, theoretically as well as in practice. This perspective offers opportunities to further study students' use of communicational resources in their writing.

Below, the contribution of this thesis is discussed in relation to the previous research and theory that has been presented above.

The concept of communication

Authoritative texts are texts that are thought to influence the practices from which they arise. The first study examined how the concept of communication is described in authoritative texts which were deemed important for mathematics education internationally. The findings indicate that the cybernetic conceptualization of communication as a relatively unproblematic transfer of ideas from one system [human] to another exerts a strong influence on the way communication is viewed in mathematics education. In this view, communication in mathematics becomes not only independent of the addressee but also transparent. Communication itself does not pose a problem, either for teachers or for students. This is contradicted by research that suggests that the communicational aspects of writing, such as semantic structure, vocabulary and mathematical symbolism, may in fact be more problematic for students than calculations and standard algorithms (Ellerton & Clarkson, 1996). A view of communication as unproblematic involves the risk that these problems become hidden or that teachers misunderstand them and see them as problems

related to understanding the mathematics involved rather than problems connected to communication.

Viewing communication as unproblematic and transparent also poses a problem in relation to the relationship between communication and knowledge. In this way the results of the first study confirm Morgan (1999) claims that although mathematics educators have widely accepted constructivist ideas in relation to how students make sense of mathematical activity, there is still a naïve understanding of communication as mere 'transmission' when it comes to assessment. In the first study, which examined communicational rules and logic, this transmission metaphor is reflected in a rule which states that communication in mathematics is independent of the addressee. This implies that meaning resides within the text, where it accurately reflects the intentions, thoughts and ideas of the author, and it is, thus, the work of an examiner to extract and interpret this meaning. In relation to evidence suggesting that students find the semantic structure and formal symbolism more difficult than the mathematical calculations, this becomes problematic. In a situation where mathematics teachers worldwide use students writing to make judgements about their level of achievement, the relationship between what a student writes and what she knows needs to be problematized. A validation of inferences made from students' writing cannot be made with a blind faith in a straightforward correspondence between text and knowledge. As is indicated in the third and fourth studies, there are a number of different ways in which students can document their problem solving, many of which have little to do with their choice of problem-solving strategy or level of achievement.

Teachers' assessment literacy

Earlier studies of teachers' skills and competence in assessment have indicated that teachers need to know more about educational assessment (Brookhart, 2001; Popham, 2009). Assessment competence or assessment literacy encompasses different skills, all of which affect the teaching and learning in an individual teacher's classroom. In making judgments about students' level of achievement, teachers draw on a number of resources that include their knowledge, experience, beliefs and expectations (Morgan & Watson, 2002). When it comes to the assessment of students' mathematical texts teachers have expectations regarding how mathematical knowledge can or should be communicated as well as preferences for

certain modes of communication. The picture on the title page of this thesis is an example of a communicational situation where the expectations and preferences of the teacher and did not match those of the student who drew the picture. The second study indicated that the teachers did have particular preferences regarding modes as they discarded the mode of image, deeming it a cognitive tool connected to a concrete strategy which represented a low level of achievement. Smith (2003) showed, in a study on young students' use of representations in problem solving, that they create their representations ad hoc to assist them in their solving of a particular problem and that this does not necessarily represent students' understanding or ability to generalize or deal with abstractions. Making valid inferences about students' knowledge or understanding from the texts that they produce is part of teachers' assessment literacy. A deeper understanding of the problematic relationship between students' texts and their knowledge would add to the validity of such inferences.

The teachers in the second study did consider students' ability to communicate in their general assessment of their texts, but in contrast to their interest in students' strategies, where they had a number of ideas on how to develop students' choices and employment alternative strategies, they had very few suggestions on how to develop students' communicational abilities. This aligns with the view of communication as something that develops un-problematically alongside other mathematical abilities. The teachers agree, however, that there may be several reasons for students' "poor" writing. They suggest that students may be unaware of what is expected of them or that they are unwilling to show faulty strategies. One teacher also suggested that a student might know how to solve the problem but not how to present her solution in writing. One way to deal with such a lack of knowledge about appropriate ways to communicate is to have teaching pay explicit attention to the form and, consequently, explicitly teach students to write (Pimm & Wagner, 2003; Solomon & O'Neill, 1998). Such teaching, however, requires that teachers have knowledge about writing and strategies for teaching this to students.

The second study suggested that teachers work in two very different modes when interpreting and assessing students' texts. In the pedagogical mode, teachers are sympathetic and spend considerable effort trying to understand what the students have done. In the assessment mode the teachers are focused on the communicational aspect of students' texts and

they are concerned with elements that are 'missing'. They frequently refer to an unarticulated 'ideal' text but they fail to agree on the particular features of such an ideal text. Most suggestions are contested and often discarded as it becomes clear to the teachers that their claims seem to be based on particular practices whose relationship to the syllabus is unclear. One example is the idea that students should state the answer to a problem by including it in a complete sentence in their texts. When asked why students should do this most teachers struggled to come up with a clear motive. Strategies to help solve the problem of students not knowing how to communicate adequately in writing were few in the teachers' discussions, and the second study thus confirms the problem described by Morgan (1998, 1999) that teachers have few tools for analyzing and developing students' writing beyond unarticulated and rather vague ideas of an ideal text.

Students' documentation of problem solving

A way of teaching that targets writing in mathematics should be based on a thorough understanding of the different aspects or elements of this writing. Writing requires attention to different questions regarding not only the context but also the skills to make meaning with the help of various semiotic resources. In the third and the fourth studies, the view of mathematical writing as the combination of mathematical language and natural language has offered the opportunity to investigate and describe the ways in which components from these different languages could be combined and integrated to construct and design an account of a problem-solving process. These studies have thus, at least in part, done what Blommaert (2013) called for: they have disassembled the mathematical writing of young students to allow for different components of writing to be distinguished. The findings illustrate a variety of ways to account for mathematical problem solving in writing, and the inventory presented is a potential tool for teachers interested in analyzing and developing their students' writing.

Earlier findings in studies of students' explanatory writing included categories: recount, summary and dialogue (Clarke et al., 1993; T. S. Craig, 2011), all of which are visible in the analysis in the third and fourth studies. Students recount their problem-solving processes, summarize their problem solving in an answer and also enter into dialogue with their teachers on some occasions. In contrast to the explanatory writing that

was examined in earlier studies, my studies were concerned with the documentation of a problem-solving activity. An analysis of the documentation of mathematical problem solving has to take into account that the students are solving the problem and documenting their process at the same time. It has been suggested earlier that these two processes often are concurrent in school mathematics (Morgan, 1998). Recounting may be seen as something that is done after the activity it is supposed to account for. The correspondence between the documentation of a problem-solving process and the problem-solving process itself is problematic (Love, 1988). The documentation of a process may serve cognitive purposes such as processing information, recording data, allowing for exploration or manipulation, or monitoring and assessing one's own progress. The documentation may also serve the purpose of providing someone with information on how the process was played out, in which case it can be said to have a social function. It has been shown earlier that students use their writing during problem solving for all these purposes (Stylianou, 2011). In a mathematical text, which is separated in time and space from the problem-solving activity it was supposed to document, the separation of the process of solving the problem and the description of the process is difficult to discern.

Different communicational choices

Separating a process from the description of it is directly parallel to the difficulty of separating form from content in any kind of communication. Is it at all fruitful or even possible to distinguish the students' communicational ability from their mathematical understanding? The findings in the third and fourth studies indicate diversity regarding nearly every aspect of documenting and communicating a problem-solving process. That the communicational choices which students face in their design of a text documenting a problem-solving activity are numerous is illustrated by the fact that students have many different ways of organizing their texts and integrating mathematical and everyday language. The great diversity indicates that students seem to have very different ideas regarding aspects such as how, what, for whom and why they are writing. It could be argued that they perceive the rules of the particular language game in which they are involved in different ways (Wittgenstein, 1953/1986; Öhman, 2006). It would be natural to expect variation in relation to the age of the students or the student groups to which they belong. It turns out however that diversity can be found across classes as well as age groups. The older the

students are, the more elements can be found in their texts that facilitate communication with a reader who is unfamiliar with the context. It is, however, not uncommon for younger students to exhibit the same kind of reader focus or for older students to lack it. Age and group norms do not predict students' communicational choices.

Several traditions in communication theory (see article 1) take an interest in the social aspect of communication and they assume that the aims, experiences and behaviors of actors in communication will significantly affect any outcome (Littlejohn & Foss, 2011). The findings of the first study suggested that the way communication is viewed in mathematics education is similar to way it is presented in the cybernetic tradition, where communication can be seen as the transferring of information and where messages are seen as input or output between systems. This, in turn, suggests that mathematical communication is seen as independent of addressee. While the students in the fourth study explicitly state that they have a particular reader in mind when they design their texts, the sample of texts in the third study suggest a number of different ways to attend to the question of for whom they are writing. In both studies, findings indicate that students use meta-text and different semiotic resources in different ways to facilitate the reading of their texts by an imaginary reader. The diversity can be seen as evidence of a varied understanding of the characteristics of this reader; thus, the students' writing is not independent of addressee.

If the diversity in student texts, with regard to what, how, for whom and why students write, is taken as evidence of the multitude of communicational choices that students face, this indicates that a distinction between the form of mathematical communication and content of such communication is in fact possible. The many different ways to organize a text, the various modes from which to choose and the various resources on which the students draw to make meaning suggests that writing can be thought of as an object of learning that is at least partly separate from the mathematical content. Given that the students in the studies all solved the same type of problem and still designed their texts in so many different ways, there is reason to believe that the communicational choices that students make are not entirely connected to the mathematical content. It is possible that many students are unaware of the choices they make, but it can be argued that even implicit assumptions and ideas regarding how to successfully express an idea or a process affect the design of a text. Being una-

ware of why one is choosing to communicate in a certain way does not mean that one is not making choices. It is their understanding of the communicative situation that makes individuals choose to draw instead of describing the same thing with words, even in cases where these individuals are unable to identify and explain their reasoning. That text making requires decisions is fundamental in social semiotics where text making is said to require semiotic work (Kress, 2010b). Mavers (2009) has suggested that even young students' text-making constitutes principled engagement in meaning making where their choices and ensembles of choices give different shapes to knowledge. When young students are viewed as competent text makers who make choices, writing can advantageously be turned into an object of learning where implicit assumptions can be identified by students and teachers and where students' communicational ability can develop.

Communication separate from content

The findings of the first study indicated that the frameworks investigated operate with a rule that states that the mathematical language with which students communicate in mathematics is a preconceived language. Such an understanding of mathematical language indicates a referential view of mathematics in which mathematical objects are thought of as having some kind of referent outside of language (Dörfler, 2002). Dörfler argued for the possibility that this is, in fact, a widespread tacit belief among many mathematicians. Dörfler also argued that this view leads mathematicians and educators to adopt a perspective in which mathematical objects are already there just waiting to be grasped by students. This can be seen as having been partly confirmed by the first study. Such a perspective can be seen to separate mathematics and mathematical knowledge from communication which leads to a description of learning mathematics as a process mediated by communication (Lampert & Cobb, 2003; Morgan et al., 2014). This thesis has argued for an empirical separation of content and form, that the ability to communicate [in writing] in mathematics can be distinguished from the ability to successfully employ various strategies in mathematical problem solving. Such a separation, however, is not based any referential view of mathematical objects. It is based on acknowledging that mathematical communication involves more than the use of formal mathematical expressions or references to mathematical objects such as numbers or geometrical shapes. Mathematical writing involves mathematical as well as natural language, and it is the integration of these two lan-

guages that make such writing coherent and communicative (Sfard et al., 1998; Steenrod et al., 1973). Such a perspective may be seen as focused on the communicational, rather than the mathematical, aspect of mathematical communication, and interest is directed towards the various ways in which writing can integrate mathematical and natural, everyday language in order to adequately communicate processes, ideas and understandings. From a communicative perspective, I argue that the documentation of mathematical problem solving offers a number of different ways of organizing a text, explaining various elements, accounting for context, recounting a process and stating a result, as has been suggested in the third study. It can be argued that the communicative choices that students make in their documenting of a problem-solving activity have nothing to do with mathematics at all and that this, therefore, is of no interest to mathematics education. The findings of the second study, along with previous studies on teachers' evaluation of students' mathematical texts (see for example Morgan, 1998), however, indicate that teachers do take the organization of a text, the inclusion of different elements, the choice of mode and the different ways of stating an answer as signs of students' level of achievement in mathematics. This might be partly explained by the fact that people tend to view writing as a single object that is evaluated in its totality (Blommaert, 2013). The communicativeness of a text thus affects any evaluation of its content. That the same content, or message, can be presented in different ways and subsequently lead to different interpretations is acknowledged in the rhetorical tradition in communication theory. This tradition asserts that the ability to communicate is an ability that can be cultivated and mastered through critical study (R. T. Craig & Muller, 2007). By acknowledging that mathematical communication integrates mathematical language and natural language, teachers and students have an opportunity to cultivate such ability.

To conclude, this thesis set out to examine and problematize students' writing in school mathematics and the various understandings of the relationship between students' written communication and their achievement. The theoretical and empirical problem of the thesis concerns the relationship between writing and knowledge. The findings suggest that students' writing as a source from which teachers make inferences regarding students' mathematical knowledge is problematic. I argue that from a communicational perspective the ability to communicate [in writing] in mathematics can and should be distinguished from other mathematical abilities.

By acknowledging that mathematical communication integrates mathematical language and natural language, teachers have an opportunity to turn writing in mathematics into an object of learning. This offers teachers the potential to add to their assessment literacy and offers students the potential to develop their communicational ability in order to write in a way that better reflects their mathematical knowledge.

Svensk sammanfattning

Avhandlingen handlar om unga elevers skrivande i matematik och det sätt på vilket detta skrivande kan designas, tolkas och förstås. Elevers kommunikation kan fungera som ett verktyg för lärare i bedömningen av elevers kunskaper och förmågor. Tidigare forskning har visat att matematiklärare i sin bedömning av elevers kunskaper betraktar tolkningen av elevers skrivande som något oproblematiskt och det centrala problemet, teoretisk såväl som empiriskt, i avhandlingen utgörs av denna oproblematiserade relation mellan det elever kommunicerar och deras kunnande eller förmåga. Tidigare forskning har också visat att skrivande i matematik kan vara svårare för elever att bemästra och förstå än det matematiska innehållet. Annan forskning som handlar om lärares bedömarkompetens tyder på att lärare behöver fördjupa sina kunskaper om bedömning. Syftet med avhandlingen är därför att undersöka och problematisera både elevers skrivande i matematik och olika förståelser av relationen mellan kommunikation och kunskap. Avhandlingen bygger på tanken att kommunikation och språk är centrala för undervisning, lärande och bedömning i matematik men också att kommunikation i matematik omfattar både matematiska uttryck och vardagligt språk. Vad kommunikation är, vad som räknas som kommunikation och vad som kännetecknar ”god” kommunikation kan förstås på olika sätt vilket får konsekvenser för hur och till vad kommunikation används i utbildningssammanhang. I en tid av ökat fokus på bedömning och utvärdering av utbildningsresultat behöver lärare djup kunskap om olika aspekter på elevers kommunikation, dels för att kunna hjälpa dem att utveckla sin förmåga att kommunicera och dels för att kunna göra en rättvis bedömning av det kunnande och den förmåga som elevers kommunikation kan visa på. I syfte att bidra till denna fördjupade kunskap har avhandling tagit ett brett grepp och undersöker olika aspekter av elevers kommunikation i matematik. Avhandlingen består av fyra delstudier som undersöker hur begreppet kommunikation beskrivs i auktoritativa texter, hur lärare ser på elevers skrivande, hur elever använder olika resurser i sin design av lösningar på matematiska problem samt eventuella skillnader i elevers digitala och analoga skrivpraktik.

I den första delstudien undersöktes tre internationella kompetensramverk i matematik i syfte att identifiera den kommunikationslogik som uttrycks. De tre ramverk som undersöktes var *PISA 2012 Mathematics Framework* (OECD, 2013), *Singapore Mathematics Framework* (MES, 2012) och den

amerikanska *Common Core Standards for Mathematical Practice* (NGACBP, 2010). Dessa valdes för att de internationellt utgör inflytelserika ramverk i matematikdidaktik samt för att de delvis vänder sig till olika mottagare. Delstudiens resultat pekar mot att ramverken sammantaget ger uttryck för en syn på kommunikation i matematik som något som är oproblematiskt och transparent. Resultaten pekar också på att ramverken bygger på tanken att matematisk kommunikation är oberoende av mottagare samt att det matematiska språket är ett fixt och färdigt språk med egenskaper som formellt, tekniskt, precist och tydligt. En möjlig konsekvens av detta är att problematiska aspekter av kommunikation osynliggörs och att en enkel och oproblematisk relation mellan skrivande och kunskap tas för given. Under sådana förutsättningar riskerar elevers eventuella oförmåga att genomskåda dolda och oartikulerade kommunikativa normer att skymma deras eventuella matematiska kunskaper. Det sätt på vilket kommunikation betraktas i auktoritativa texter som de undersökta ramverken får betydelse för hur lärare uppfattar och använder elevers matematiska kommunikation i sin bedömning. I delstudie två flyttades därför fokus till lärare och deras syn på elevers skrivande.

I den andra delstudien intervjuades 19 lärare i 5 olika grupper i syfte att undersöka hur de i grupp talade om och förstod elevers skriftliga lösningar av ett matematiskt problem. Lösningarna hade tidigare samlats in i samband med problemlösningssituationer i två fjärdeklasser. Lärarna intervjuades i grupper och fokus riktades i analysen på den gemensamma diskussionen snarare än de enskilda lärarnas uppfattningar. Utgångspunkten för analysen var att lärare, liksom andra yrkesgrupper, är en del av en praktik där många uppfattningar, tolkningar och värderingar delas av deltagarna. Gruppdiskussioner erbjöd då möjligheter att identifiera en del av dessa uppfattningar. Resultaten pekar på att de intervjuade lärarna, i sina diskussioner om bedömningen av elevernas lösningar, pendlar mellan två förhållningssätt: det pedagogiska och det bedömningsinriktade. Det bedömningsinriktade förhållningssättet fokuserar på elevers kommunikativa val, exempelvis hur de valt att svara på frågan eller på vilket sätt de beskrivit sin process. Det pedagogiska förhållningssättet är inriktat på att förstå hur eleverna tänkt och resonerat samt att identifiera olika lösningsstrategier. I det pedagogiska förhållningssättet tolkar lärarna elevernas kommunikation med en välvilja inriktad på förståelse. Lärarna ser det som sitt jobb att förstå vad eleverna gjort så att de kan hjälpa dem att utveckla mer effektiva och hållbara problemlösningsstrategier. Det innebär att de

lägger stor möda på att förstå även när elevens text innehåller väldigt lite information. När lärarna diskuterar elevernas lösningar utifrån det bedömningsinriktade förhållningssättet fokuserar de nästan uteslutande på hur eleverna kommunicerat sina lösningar och i synnerhet på brister i elevernas redovisande. Ett exempel på hur lärare pendlar mellan dessa två förhållningssätt är uppfattningen att något eleven skrivit visserligen är fullt begripligt men inte tydligt kommunicerat. När lärarna diskuterar elevernas texter utifrån ett bedömningsinriktat förhållningssätt är de överens om att eleverna borde kunna utveckla sin förmåga att kommunicera skriftligt i matematik, men till skillnad från det pedagogiska förhållningssättet där de har många förslag på hur undervisningen ska hjälpa eleverna, så har de få förslag på hur de ska hjälpa eleverna att utveckla sin skriftliga kommunikation. De refererar återkommande till en ”bra” lösning men kan inte enas om tydliga exempel på vad som utmärker en sådan. För att lärare ska kunna hjälpa elever att kommunicera skriftligt på ett kompetent och effektivt sätt måste de ha fördjupade kunskaper om olika sätt att skriva samt underlag för diskussioner om hur dessa olika sätt kan värderas. De två sista delstudierna ägnas därför åt att undersöka elevers skrivande i syfte att bidra till en denna fördjupade kunskap.

I den tredje delstudien undersöktes elevers matematiska skrivande genom en analys av 519 elevlösningar producerade av elever mellan 9 och 12 år i samband med problemlösningsarbete under matematiklektioner. Syftet med studien var att analysera elevers olika kommunikativa val samt de resurser de tar i anspråk i sitt skrivande. Lösningarna samlades in från tio lärare på åtta olika skolor och totalt sexton klasser. Alla problem var av samma typ, de handlade alla om att distribuera något på olika sätt, till exempel ben på olika djur eller hjul på olika fordon. Lärarna, från vilka lösningarna samlades in, hade instruerats att skapa problemlösningssituationer som skiljde sig så lite som möjligt från det de normalt gjorde tillsammans med sina elever samt att omformulera eller modifiera de olika problemen på det sätt som de själva tyckte passade i respektive elevgrupp. Lösningarna analyserades med utgångspunkt i multimodal diskursanalys med målet att skapa en katalog av exempel på de olika kommunikativa val elever gör samt de resurser de använder. Resultatet av analysen visar att elever har många olika sätt att dokumentera sin problemlösning. Elever använder bilder, ord, siffror, symboler och layout för att visa, förklara och strukturera sina lösningar så att de kan läsas av andra. Även bland lösningar från elever i samma klass, där man kan utgå från att eleverna fått

samma instruktioner och där någon typ av norm utvecklats genom tidigare undervisning, var variationen stor. Resultaten indikerar att äldre elever lägger mer tid och energi på att underlätta för en utomstående läsare utan kunskap om kontexten, men den typen av läsarfokus finns också bland yngre elever. Analysen har inte fokuserat på de matematiska strategier som eleverna valt eller på den förståelse som dessa strategier skulle kunna tyda på men trots detta kan man se att sambandet mellan matematisk problemlösningsförmåga och kommunikativ förmåga inte är entydigt. En välstrukturerad lösning som är rik på information kan visa på en potentiellt begränsad matematisk förståelse, medan en ostrukturerad och knappt läsbar lösning kan tyda på en mycket god förståelse. Nedan återfinns en tabell över de olika resurser som eleverna använt samt en förteckning över vad de använts till.

Tabell 3 *Sammanfattning av de olika modes som eleverna använt samt vad de använts till*

	BILDER	**ORD**	**SIFFROR**	**SYMBOLER**	**LAYOUT**
Titel på lösningen		X			
Ange villkoren för problemet	X	X	X	X	
Visa problemlösnings-processen	X	X	X	X	
Illustration	X				
Förklara olika element eller ange enheter		X	X		
Meta-text, strukturerande av lösningen	X	X	X		X
Ange ett svar på frågan	X	X	X	X	

En tabell som den ovan kan, även om den endast omfattar delar av elevers möjliga kommunikativa val, fungera som ett verktyg för lärare i utvecklingen av undervisningsstrategier för att hjälpa eleverna att designa bättre lösningar. Den andra delstudien indikerade att sådana strategier till stora delar saknades i lärargrupperna.

I den fjärde delstudien undersöktes sju elevers dokumentation av problemlösning producerade med hjälp av papper och penna och med hjälp av en interaktiv skrivtavla. Av de 28 elevlösningar, på samma typ av problem som i delstudie tre, som samlades in, var hälften producerade analogt med papper och penna, och hälften digitalt med hjälp av en interaktiv skrivtavla. De sju eleverna intervjuades var och en i samband med den sista problemlösningsaktiviteten. Liksom i delstudie tre pekar resultaten på att elever har tillgång till och använder en mängd olika resurser i sitt dokumenterande av matematisk problemlösning och att dessa skiljer sig åt, om än i relativt liten utsträckning, mellan den analoga och digitala situationen. En tydlig skillnad mellan det digitala och det analoga skrivandet är användandet av färg. Eleverna använder färg i sitt digitala skrivande bland annat för att skilja de olika exemplen åt. Trots att de har tillgång till färgpennor när de skriver på papper är det ingen av eleverna som använder färg. En annan skillnad är avsaknaden av linearitet i flera av de digitala lösningarna. Eleverna börjar i mitten och utvidgar sin lösning åt alla håll. De lösningar som skapats med papper och penna har en tydlig linearitet uppifrån och ned samt från vänster till höger vilket hjälper en läsare att förstå i vilken ordning olika beräkningar gjorts. I intervjuer berättar eleverna att de tänker sig en mottagare för texten för vilken de måste förklara olika saker, men det finns en tydlig skillnad mellan de digitala och analoga texterna i hur mycket information som ges till en utomstående läsare. Elevernas vanliga användande av digitalt skrivande sker i princip aldrig enskilt utan utgör en del av en situation där skrivande kan kompletteras med verbala förklaringar. Eleverna anpassade sig delvis till det faktum att de saknade möjlighet att verbalt komplettera sina lösningar men en tydlig skillnad mellan de digitala och analoga lösningarna återstår. Sammantaget tyder delstudie tre och fyra på att elever förhåller sig sinsemellan olika till frågor kring vad, hur, med vem samt varför de kommunicerar.

Avhandlingens resultat bekräftar den tidigare forskning som pekat på det problematiska i att oreflekterat använda elevers skrivande i matematik för att dra slutsatser om deras olika matematiska kompetenser. Avhandlingen visar på möjligheten för lärare i matematik att göra kommunikation och skrivande i sig till ett undervisningsobjekt, bland annat genom att skilja mellan förmågan att lösa matematiska problem och förmågan att beskriva sin lösningsprocess samt att förutsätta att kommunikation i matematik

omfattar både matematiska uttrycksformer och vardagligt språk. På så sätt ges elever möjlighet att utveckla sin förmåga att uttrycka sig på ett sätt som bättre speglar deras kunnande samtidigt som lärare ges verktyg att utveckla sin bedömningskompetens.

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